Lifecycle Investing in Stochastic Environments

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Abstract

This paper extends Liu (2007) to capture lifecycle risks beyond investment returns and studies the conditions under which lifecycle paths become stationary, establishing a basis to compare lifepaths across generations. Modelling risky income, stochastic consumption prices and risky returns together makes it possible to capture common factors embedded into these processes through a shared state process. As in Liu (2007), closed form solutions are available when stochastic processes follow certain quadratic dynamics including the models with stochastic market price of risk and stochastic volatility with market price of risk proportional to volatility.

Investment returns together with income and consumption price changes are important sources of uncertainty when taking consumption and investment decisions. Moreover, there is evidence that these components are related beyond simple correlations, as argued by scholars in the economic and finance literature. For instance, there is evidence pointing towards house prices, rent prices and income cointegration (Malpezzi, 1999; Meen, 2002; Gallin, 2008; Holly, Pesaran, & Yamagata, 2010). Another case is cointegration between income and dividends as argued by Benzoni, Collin-Dufresne, and Goldstein (2007). To capture these type of relationships in a comprehensive manner, lifecycle models need to incorporate shared state processes among asset prices, income and consumption prices.

There exist lifecycle investment models capturing each of the features described above, but not all of them together. The earliest continuous time portfolio optimization problem is Merton (1971), which provides closed form optimal consumption and investment policies for investors with constant relative risk aversion and human capital. Kim and Omberg (1996) and Wachter (2002) solved a similar problem under a stochastic but partly predictable risk premium embedded into the state process. Liu (2007) generalized this framework to investment and state dynamics that follow quadratic processes, including stochastic volatility based on the Heston model or the previous stochastic risk premium model. Benzoni et al. (2007) develop a lifecycle model with return predictability by means of a stationary income-dividend ratio embedded in the state process, and solve the problem numerically.

Stochastic consumption prices and Cobb-Douglas consumption bundle aggregators have been widely used to model housing services. Early works like Damgaard, Fuglsbjerg, and

Munk (2003), Cocco (2005) and Yao and Zhang (2005) were solved numerically and focused on modelling housing frictions such as indivisibility and illiquidity. Kraft and Munk (2011) managed to obtain closed form solutions after removing frictions.

The main contribution of this paper is extending the consumption and investment problem of Liu (2007) to incorporate risky income and stochastic consumption prices sharing a common state process while maintaining tractability. These new components share the state process with asset prices, making it possible to capture more complicated relations such as cointegration. Like in Liu (2007), exact solutions to quadratic cases are general enough to include stochastic market price of risk and stochastic volatility with market price of risk proportional to volatility. Additionally I extend the separation theorem to derive exact solutions when state process innovations are correlated between state components.

The consumption and investment problems faced by individuals share a common structure regardless of parameters or time references. I find that, under some stationarity conditions, a reference income process becomes a natural numeraire for individuals whose consumption is exposed mainly to labor price risk or wanting to keep up with the Joneses. One may then prefer to measure savings as years of average income saved and consumption rates in proportion to average income.

In this paper I provide a comprehensive modelling framework, including an auxiliary asset pricing model and tools to measure welfare changes under alternative investment strategies for evaluating counterfactual scenarios. The asset pricing model considers investment assets, like stocks, housing or bonds, as claims to a stochastic payoff stream. This is useful to structurally relate asset prices with consumption prices or the income process, and to understand the implications that those relations have on asset returns and common financial indicators under rational assumptions. When imposing quadratic and affine structures, the solution to associated pricing PDEs is very similar to that of zero-coupon bond prices with quadratic term structures (Ahn, Dittmar, & Gallant, 2002), but generalized to stochastic payoff processes instead of a deterministic unitary payoff.

I apply this model to a lifecycle investment problem with housing and income cointegration in García (2026). That companion paper captures housing, income and even stocks cointegration in a tractable lifecycle model exploiting the tools that I develop here. However it is more of an empirical paper, since it estimates model parameters and evaluates the impact that housing and income cointegration have on investment, consumption and welfare.

The structure of this paper is organized as follows. Section 1 lays down financial markets and the asset pricing model. Section 2 solves the lifecycle investment problem and in Section 3 I study how to measure welfare changes under alternative investment policies. Then Section 4 analyzes the conditions under which lifecycle paths become stationary and comparable across generations. Section 5 explains how to solve PDEs encountered in the previous sections. Applications of this model are described in Section 6. Finally Section 7 provides some concluding remarks. A summary of notation is provided in Section A.1.

1 Asset prices and returns

Many investment assets can be understood as claims to future payoffs or to an stream of future payoffs. For instance bonds are claims to future coupon payments and the principal, stock shares are claims to future earning distribution payments and houses are claims to future rent payments or dwelling rights, which are also valuable. Typically, investors purchase an asset and collect payoffs until maturity or until they sell again the asset. Obtained returns depend on the purchase price, on the value of collected payoffs and on the selling price. In turn, the selling price of an asset depends on its remaining payoff claims and on market expectations. These payoff processes may be related to one another or to state variables X_t, and capturing these relations may be critical to understand its return dynamics.

I begin this section explaining the model for financial markets. Then I derive the price to a future risky payoff, the price to a stream of future risky payoffs and analyze the properties of price multipliers. Afterwards I derive and analyze return dynamics.

In this economy, there is a state vector process X_t of size n_X driven by standard Brownian motion vector $Z_{X,t}$ of size n_{Z_X} where the drift vector $\mu_{X,t} \coloneqq \mu_X(t, X_t)$ and diffusion matrix $\Sigma_{X,t} \coloneqq \Sigma_X(t, X_t)$ may also depend deterministically on state X_t and time

$$\mathrm{d}X_t = \mu_{X,t}\,\mathrm{d}t + \Sigma_{X,t}\,\mathrm{d}Z_{X,t}\,. \tag{1}$$

The investment product universe available to individuals comprises the instantaneous risk-free rate r_t and risky assets with cumulative returns given by the vector process A_t of size n_A with strictly positive components

$$\frac{\mathrm{d}A_t}{A_t} = \mu_{A,t}\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\,. \tag{2}$$

The risk free rate $r_t \coloneqq r(t, X_t)$ as well as the assets drift vector $\mu_{A,t} \coloneqq \mu_A(t, X_t)$ and diffusion matrix $\Sigma_{A,t} \coloneqq \Sigma_A(t, X_t)$ are deterministic functions of state X_t and time. Returns are driven by standard Brownian motion vector $Z_{A,t}$ of size n_{Z_A} and it is assumed that the inverse covariance matrix $\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}$ exists almost surely. There exists a market price of risk vector $\Lambda_t \coloneqq \Lambda(t, X_t)$ of size n_{Z_A}, which may depend on state X_t and time, satisfying the no arbitrage constraint

$$\Sigma_{A,t}\Lambda_t = \mu_{A,t} - r_t\mathbf{1} \tag{3}$$

and the pricing kernel $K_t \coloneqq K(t, X_t)$ of this economy has dynamics

$$\frac{\mathrm{d}K_t}{K_t} = -\,r_t\,\mathrm{d}t - \Lambda_t^\intercal\,\mathrm{d}Z_{A,t}\,.$$

Instantaneous correlation matrices between Brownian motion vectors are denoted by $\rho_{\square\square}$ and they correspond to the instantaneous rate of change of quadratic covariation, e.g. $\mathrm{d}[Z_{X_t}, Z_{A,t}] = \rho_{XA,t}\,\mathrm{d}t$. In this case, the matrix process $\rho_{XA,t} \coloneqq \rho_{XA}(t, X_t)$ of size $n_{Z_X} \times n_{Z_A}$ is assumed to depend only on time and state X_t.

Brownian motion vectors belong to a filtered probability space satisfying the usual conditions and control variables are assumed to be adapted to this filtration $\mathcal{F}_t$, ensuring that decisions at time t consider information available only up to that time. Partial derivatives are expressed using operators and may omit secondary function arguments, for instance $\partial_t f_t = \partial_t f(t, X_t)$ refers to the partial time derivative and $\partial_X f_t = \partial_X f(t, X_t)$ to the partial state derivative. Also, I denote the orthogonal projection matrix of $\Sigma_{A,t}^\intercal$ capturing the degree of market completeness with

$$\mathcal{P}_{\Sigma_{A,t}^\intercal} \coloneqq \Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}.$$

Definition 1 (Replicability). *Consider a scalar stochastic process φ_t with dynamics that can be decomposed in terms of a drift $\mu_{\varphi,t} := \mu_\varphi(t, X_t, H_t)$ and a diffusion term driven by standard Brownian motion vector $Z_{\varphi,t}$ of size n_{Z_φ} with diffusion matrix $\Sigma_{\varphi,t} := \Sigma_\varphi(t, X_t, H_t)$*

$$\mathrm{d}\varphi_t = \mu_{\varphi,t}\,\mathrm{d}t + \Sigma_{\varphi,t}\,\mathrm{d}Z_{\varphi,t}\,.$$

Parameters may refer to some adapted vector process H_t, and the instantaneous correlation between Brownian motions $Z_{\varphi,t}$ and $Z_{A,t}$ is denoted by $\rho_{\varphi A,t} := \rho_{\varphi A}(t, X_t)$. Process φ_t is said to be spanned by asset risk factors if process diffusion satisfies

$$\Sigma_{\varphi,t}\,\mathrm{d}Z_{\varphi,t} = \Sigma_{\varphi,t}\rho_{\varphi A,t}\,\mathrm{d}Z_{A,t}\,. \tag{4}$$

Additionally process φ_t is said to be replicable if there exists investment strategy vector $\xi_{\varphi,t} := \xi_\varphi(t, X_t, H_t)$ that exactly reproduces the diffusion term

$$\Sigma_{\varphi,t}\,\mathrm{d}Z_{\varphi,t} = \xi_{\varphi,t}^\intercal \Sigma_{A,t}\,\mathrm{d}Z_{A,t}$$

which after substituting its projection $\xi_{\varphi,t} = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{\varphi A,t}^\intercal\Sigma_{\varphi,t}^\intercal$ becomes

$$\Sigma_{\varphi,t}\,\mathrm{d}Z_{\varphi,t} = \Sigma_{\varphi,t}\rho_{\varphi A,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\,\mathrm{d}Z_{A,t}\,. \tag{5}$$

Moreover process φ_t is said to be self-financing replicable if the replication strategy satisfies

$$\mathrm{d}\varphi_t \leq \varphi_t r_t\,\mathrm{d}t + \xi_{\varphi,t}^\intercal\left(\frac{\mathrm{d}A_t}{A_t} - r_t\mathbf{1}\,\mathrm{d}t\right). \tag{6}$$

Lemma 1. *A replicable process implies that it is spanned by asset risk factors.*

Proof. Computing the quadratic covariation on both sides of (5) with respect to $Z_{A,t}$ yields the equivalence below, which can be replaced in (5) to obtain (4).

$$\Sigma_{\varphi,t}\rho_{\varphi A,t} = \Sigma_{\varphi,t}\rho_{\varphi A,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}$$

□

Lemma 2. *The quadratic variation of processes on each side of (4) are equal if and only if process φ_t is spanned by asset risk factors $Z_{A,t}$. Additionally, the quadratic variation of processes on each side of (5) are equal if and only if φ_t is replicable.*

Proof. See Section A.2. □

Having establish the setting, now I proceed to derive price of claims to future payoffs.

1.1 Price of a payoff claim

Consider the strictly positive scalar payoff process Q_t below with time and state dependent parameters $\mu_{Q,t} := \mu_Q(t, X_t), \Sigma_{Q,t} := \Sigma_Q(t, X_t)$. It is driven by standard Brownian motion vector $Z_{Q,t}$ with time and state dependent instantaneous correlation matrices $\rho_{QA,t} := \rho_{QA}(t, X_t)$, $\rho_{XQ,t} := \rho_{XQ}(t, X_t)$ to previously defined Brownian motion vectors.

$$\mathrm{d}Q_t = Q_t\left(\mu_{Q,t}\,\mathrm{d}t + \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t}\right) \tag{7}$$

Lemma 3 (Terminal payoff price). *The price $\Omega(t, X_t, Q_t; T)$ at time t of an uncertain payoff Q_T to be received at time T is*

$$\Omega(t, X_t, Q_t; T) = \mathbb{E}_t\left[\frac{K_T}{K_t} Q_T\right] = Q_t \tilde{\Omega}(t, X_t; T)$$

where price multiplier $\tilde{\Omega}(t, X_t; T)$ solves the following partial differential equation (PDE)

$$\begin{aligned} 0 =& \partial_t \tilde{\Omega}_t + \tilde{\Omega}(t, X_t; T)\left(\mu_{Q,t} - r_t - \Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right) \\ &+ (\partial_X \tilde{\Omega}_t)^\intercal \left(\mu_{X,t} + \Sigma_{X,t}\left(\rho_{XQ,t}\Sigma_{Q,t}^\intercal - \rho_{XA,t}\Lambda_t\right)\right) + \frac{1}{2}\operatorname{tr}\left(\Sigma_{X,t}\Sigma_{X,t}^\intercal \partial_{XX^\intercal}\tilde{\Omega}_t\right) \end{aligned} \tag{8}$$

with boundary condition $\tilde{\Omega}(T, X_t; T) = 1$, assuming [1] *that the price process is spanned by asset risk factors $Z_{A,t}$. When asset price $\Omega(t, X_t, Q_t; T)$ is replicable (Definition 1), the replicating strategy is*

$$\pi_{\Omega,t} = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t} \frac{\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Omega_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Omega_t}{\Omega(t, X_t, Q_t; T)}$$

producing the following self-financing portfolio dynamics

$$\frac{\mathrm{d}\Omega_t}{\Omega_t} = r_t \,\mathrm{d}t + \pi_{\Omega,t}^\intercal \left(\frac{\mathrm{d}A_t}{A_t} - r_t \mathbf{1}\,\mathrm{d}t\right).$$

Proof. See Section A.3 for the price and Section A.4 for the replicating strategy. □

On the technical side, the linear PDE (8) for price multiplier $\tilde{\Omega}(t, X_t; T)$ is an instance of the semi-linear PDE for $g(t, X_t)$ described in Definition 3 and studied in Section 5. The explicit relationship between them and some additional remarks can be found at the end of Section A.3.

The price $\Omega(t, X_t, Q_t; T)$ of a claim to a future payoff in Lemma 3 is the current value of the payoff process times a state dependent price multiplier $\tilde{\Omega}(t, X_t; T)$. The first line of (8) shows that price multipliers incorporate expectations about payoff trends, the risk free rate discount and a discount for payoff risk at the market price of risk. On the second line we can see that price multipliers also incorporate expectations about state trends, their relation to payoffs and a discount for state risk at the market price of risk.

Lemma 4 (Payoff stream price). *The price $\Upsilon(t, X_t, Q_t; T)$ at time t of an uncertain payoff stream Q_t with dynamics* (7) *from time t to T in terms of $\tilde{\Omega}$ from Lemma 3 is*

$$\Upsilon(t, X_t, Q_t; T) = \int_t^T \Omega(t, X_t, Q_t; s)\,\mathrm{d}s = Q_t \int_t^T \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s\,. \tag{9}$$

When asset price $\Upsilon(t, X_t, Q_t; T)$ is replicable (Definition 1), the replicating strategy is

$$\pi_{\Upsilon,t} = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t} \frac{\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Upsilon_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Upsilon_t}{\Upsilon(t, X_t, Q_t; T)}.$$

and self-financing replicating portfolio dynamics allow distributing a continuous payoff stream Q_t

$$\mathrm{d}\Upsilon_t = -Q_t \,\mathrm{d}t + \Upsilon_t \left(r_t \,\mathrm{d}t + \pi_{\Upsilon,t}^\intercal \left(\frac{\mathrm{d}A_t}{A_t} - r_t \mathbf{1}\,\mathrm{d}t\right)\right).$$

[1] The price is algebraically the same without this assumption, however one could argue that unspanned risk factors have an undefined price of risk. Without loss of generality, one can explicitly enlarge $Z_{A,t}$ with those extra risk factors and assign them a zero market price of risk to remove ambiguity.

Proof. See Section A.5 for the price and Section A.6 for the replication strategy. □

The price $\Upsilon(t, X_t, Q_t; T)$ of a payoff stream is the integral of payoff prices over maturities up to time T. This payoff stream price can be decomposed into the current value of the payoff process Q_t and a state dependent price multiplier integrating the previous $\tilde{\Omega}(t, X_t; s)$ payoff price multipliers from Lemma 3 over maturities $s \in [t, T]$.

1.2 Price multiplier stationarity

Asset pricing models typically capture some stylized facts about financial markets or some hypothesized relationship, while satisfying some restrictions, rational assumptions and maintaining plausibility with respect to other empirical findings. Examples of financial indicators and mechanisms include mean reversion of interest rates (Vasicek, 1977) or the dividend yield (Campbell & Shiller, 1988). With respect to model restrictions, stationarity of price multipliers is often assumed to rule out rational bubbles in infinite horizons (Blanchard & Watson, 1982).

Many financial indicators are stated in terms of price multipliers or their inverse ratios. For instance, common stock market indicators include price-to-earnings, price-to-dividend or the dividend yield. The bond market has the price-to-coupon or the current yield ratio. With respect to real estate, indicators include the price-to-rent ratio or the rent yield. These financial indicators are usually constructed keeping some index characteristics fixed but allowing changes in the underlying constituents. U.S. treasury indicators show the current yield ratio for 2-year, 5-year, 10-year and 30-year constant maturities. Stock market indicators typically show the price-to-earnings for the largest firms within a sector by market capitalization, and some housing market indicators capture the average price-to-rent ratio for house sales occurring within a given geographical area.

Lemma 5 (Price multiplier stationarity)**.** *Suppose that market parameters only depend on state vector process X_t but not on time. Reformulating terminal time $T = t + \tau_{T,t}$ of PDE (8) in terms of a reference time horizon $\tau_{T,t}$, characterizes a price multiplier $\bar{\Omega}(\tau_{T,t}, X_t)$ with boundary condition $\bar{\Omega}(0, X_t) = 1$ that depends on the reference time horizon $\tau_{T,t}$ and state X_t, but not directly on time. Additionally, suppose that the reference time horizon process $\tau_{T,t}$ depends only on the state process X_t or is constant. Then Lemma 3 and Lemma 4 price multipliers are pure mapping functions of the state process X_t*

$$\begin{aligned}\frac{\Omega(t, X_t, Q_t; T)}{Q_t} &= \tilde{\Omega}(t, X_t; t + \tau_{T,t}) = \bar{\Omega}(\tau_{T,t}, X_t)\\ \frac{\Upsilon(t, X_t, Q_t; T)}{Q_t} &= \int_0^{\tau_{T,t}} \tilde{\Omega}(t, X_t; t + s)\,\mathrm{d}s = \int_0^{\tau_{T,t}} \bar{\Omega}(s, X_t)\,\mathrm{d}s\end{aligned}$$

with pushforward distributions $F_{\frac{\Omega}{Q}}$,$F_{\frac{\Upsilon}{Q}}$ mapped from the state cumulative density function F_X. Suppose further that the state process is stationary, i.e. its unconditional path distribution over any interval $[t_1, t_n]$ is invariant for any time shift h

$$X_{[t_1+h, t_n+h]} \overset{d}{\sim} X_{[t_1, t_n]} \tag{10}$$

then by Kallenberg (2021, Lemma 25.1) price multiplier distributions $F_{\frac{\Omega}{Q}}$,$F_{\frac{\Upsilon}{Q}}$ are also stationary.

Lemma 5 can be used to model price multipliers based on stationarity assumptions or to capture stylized facts about financial indicators. Note that arguments in Lemma 5 extend naturally to aggregate price multipliers and composites. This can capture assets composed of a payoff stream and a final payoff, like bonds composed of coupons and a principal, or like houses composed of rent payments and a residual land value.

1.3 Return dynamics

The value of a portfolio $A_{\Omega,t}$ investing in a terminal payoff claim is solely driven by changes in the price of the claim, so $A_{\Omega,t} = \Omega(t, X_t, Q_t; T)$ with dynamics

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = \frac{\mathrm{d}\Omega(t, X_t, Q_t; T)}{\Omega(t, X_t, Q_t; T)}. \tag{11}$$

The value of a portfolio $A_{\Upsilon,t}$ investing in a payoff stream with price $\Upsilon(t, X_t, Q_t; T)$ and reinvesting intermediate payoffs in an equivalent portfolio since time t_0 corresponds to

$$A_{\Upsilon,t} = \Upsilon(t, X_t, Q_t; T) + \int_{t_0}^{t} Q_s \frac{A_{\Upsilon,t}}{A_{\Upsilon,s}} \,\mathrm{d}s\,. \tag{12}$$

Using Itô's lemma we can arrive at the return dynamics below, that account for the intermediate payoff Q_t received on top of price changes in the payoff stream claim $\Upsilon(t, X_t, Q_t; T)$ for $t \in [t_0, T)$

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = \frac{Q_t}{\Upsilon(t, X_t, Q_t; T)}\,\mathrm{d}t + \frac{\mathrm{d}\Upsilon(t, X_t, Q_t; T)}{\Upsilon(t, X_t, Q_t; T)}. \tag{13}$$

Return dynamics are more explicitly described in Lemma 6. All asset return dynamics share the risk free rate r_t, the market price of risk Λ_t and the risk factor $\mathrm{d}Z_{A,t}$. They differ only in the risk premium and risk exposure, which are both proportional to the risk loading of each asset. Expected payoff growth or state trends do not play a role, returns are only driven by discount rates.

Lemma 6 (Returns of payoff claims)**.** *Return dynamics of investing into a terminal payoff claim driven by Q_t with price $\Omega(t, X_t, Q_t; T)$ are*

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r_t\,\mathrm{d}t + \left(\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal} + \rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\frac{\partial_X\tilde{\Omega}_t}{\tilde{\Omega}(t, X_t; T)}\right)^{\intercal}(\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{A,t}) \tag{14}$$

For a payoff stream driven by Q_t with price $\Upsilon(t, X_t, Q_t; T)$, return dynamics are

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r_t\,\mathrm{d}t + \left(\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal} + \rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\frac{\int_t^T \partial_X\tilde{\Omega}(t, X_t; s)\,\mathrm{d}s}{\int_t^T \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s}\right)^{\intercal}(\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{A,t})\,. \tag{15}$$

and the compounded total return factor of the payoff stream claim for $t \in [t_0, T)$ is

$$\frac{A_{\Upsilon,t}}{A_{\Upsilon,t_0}} = \frac{\Upsilon(t, X_t, Q_t; T)}{\Upsilon(t_0, X_{t_0}, Q_{t_0}; T)} e^{\int_{t_0}^{t} \frac{Q_s}{\Upsilon(s, X_s, Q_s; T)}\mathrm{d}s}. \tag{16}$$

Proof. See Section A.7 □

2 Dynamic portfolio optimization

Suppose that an individual with initial wealth W_t at time t seeks to maximize expected CRRA utility (19) with relative risk aversion $\gamma > 0$ until time T over Cobb-Douglas consumption bundles (21). Utility derived from instantaneous consumption bundles is weighted by $\varepsilon_1 \geq 0$, and terminal consumption is weighted by $\varepsilon_2 \geq 0$. Both types of consumption are discounted with impatience rate $\delta_t \coloneqq \delta(t, X_t)$, which may depend on time and the state process X_t. The individual is endowed with an exogenous stream of labor income lasting until time $T_R \leq T$ that is captured through the payoff process Q_t with dynamics (7). Indirect utility is denoted by J_t and the problem faced by individuals is

$$
\begin{aligned}
&J(t, W_t, X_t, P_t, Q_t) \\
&= \sup_{\pi, c, \varpi \in \mathcal{A}} \mathbb{E}_t \left[\varepsilon_1 \int_t^T e^{-\int_t^s \delta_q \mathrm{d}q} u(v(\varpi_s, c_s, P_s, \tilde{\theta})) \,\mathrm{d}s + \varepsilon_2 e^{-\int_t^T \delta_q \mathrm{d}q} u(v(\varpi_T, W_T, P_T, \theta)) \right] && (17)\\
&\quad \text{s.t. } \mathrm{d}W_t = (Q_t \mathbb{1}_{t \leq T_R} - c_t) \,\mathrm{d}t + W_t \left(r_t \,\mathrm{d}t + \pi_t^\intercal \left(\frac{\mathrm{d}A_t}{A_t} - r_t \mathbf{1} \,\mathrm{d}t \right) \right) && (18)
\end{aligned}
$$

where the utility function

$$
u(x) \coloneqq \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases} \tag{19}
$$

applies over Cobb-Douglas consumption bundles

$$
v(\varpi_t, c_t, P_t, \theta) \coloneqq (c_t - c_t \varpi_t^\intercal \mathbf{1})^{1-\theta^\intercal \mathbf{1}} \prod_{i=1}^{n_P} \left(\frac{c_t \varpi_{i,t}}{P_{i,t}} \right)^{\theta_i} \quad \text{where } \varpi_t^\intercal \mathbf{1} \leq 1.
$$

At every instant, individuals choose their consumption budget $c_t \geq 0$ and allocate budget fractions $\varpi_t \geq \mathbf{0}$ to dynamically priced products. The vector of budget fractions ϖ_t is of size n_P and it is restricted to the unit simplex $\varpi_t^\intercal \mathbf{1} \leq 1$. They take this decision while observing the stochastic price vector $P_t > \mathbf{0}$ of size n_P. The Cobb-Douglas aggregator captures complementarities between different products, like housing services or recreation, and features constant returns to scale. Elasticity to dynamically priced products is given by vector $\tilde{\theta} \geq \mathbf{0}$ of size n_P for intermediate consumption and $\theta \geq \mathbf{0}$ for terminal consumption. These vectors capture the taste for each product and live in the unit simplex $\theta^\intercal \mathbf{1} \leq 1$. The remainder of the consumption budget that was not allocated to dynamically priced products, $c_t - c_t \varpi_t^\intercal \mathbf{1}$, is automatically allocated to a cash indexed product at an unitary consumption price for which the individual has a taste elasticity of $1 - \theta^\intercal \mathbf{1}$.

Wealth dynamics reflect the stochastic nature of risky investments (2), the cost of consumption and the inflow of labor income [2] Q_t. The fraction of wealth to invest in each risky asset A_t is given by decision variable π_t of size n_A. All control variables π_t, c_t, ϖ_t are subject to some admissibility restriction $\mathcal{A}$, that includes square integrability of $W_t \pi_t^\intercal \Sigma_{A,t}$ and adaptedness of all controls to the filtration $\mathcal{F}_t$. When there is no labor income process, the portfolio is self-financed.

The consumption price vector $P_t > \mathbf{0}$ is assumed to follow a stochastic process where the drift vector $\mu_{P,t} \coloneqq \mu_P(t, X_t)$ and the diffusion matrix $\Sigma_{P,t} \coloneqq \Sigma_P(t, X_t)$ may depend on state X_t

[2] The labor income process Q_t of this section has the same dynamics as the payoff process in (7), but allows the shorthand notation $Q_t = 0$ to remove this component from the model.

and time. It is driven by standard Brownian motion vector $Z_{P,t}$ with time and state dependent correlation matrices $\rho_{PA,t} \coloneqq \rho_{PA}(t, X_t)$, $\rho_{XP,t} \coloneqq \rho_{XP}(t, X_t)$, $\rho_{PQ,t} \coloneqq \rho_{PQ}(t, X_t)$ to previously defined Brownian motion vectors.

$$\frac{\mathrm{d}P_t}{P_t} = \mu_{P,t}\,\mathrm{d}t + \Sigma_{P,t}\,\mathrm{d}Z_{P,t}$$

Remark 1. *Compared to Liu (2007), my model can incorporate human capital with stochastic labor income Q_t and consumption bundles with stochastic product prices P_t. These components have been used together in Kraft and Munk (2011), but my model allows for a shared state process X_t between income Q_t, product prices P_t and risky asset returns A_t that goes beyond the risk free interest rate r_t.*

Assuming that admissibility does not place additional restrictions on consumption allocation ϖ_t, the consumption allocation decision can be optimized separately. Utility (19) is monotonically increasing in the amount of consumption bundles, therefore individuals prefer consumption allocations that maximize the amount of equivalent consumption bundles, $v(c_t, P_t, \theta) \coloneqq \sup_{\varpi_t} v(\varpi_t, c_t, P_t, \theta)$.

$$v(c_t, P_t, \theta) = \sup_{\varpi_t \in \mathbb{R}_+^{n_P}} (c_t - c_t \varpi_t^\intercal \mathbf{1})^{1-\theta^\intercal \mathbf{1}} \prod_{i=1}^{n_P} \left(\frac{c_t \varpi_{i,t}}{P_{i,t}} \right)^{\theta_i} \quad \text{such that } \varpi_t^\intercal \mathbf{1} \leq 1 \tag{20}$$

Lemma 7 (Explicit consumption bundle)**.** *The explicit solution to the consumption allocation problem* (20) *is*

$$v(c_t, P_t, \theta) = \frac{c_t}{P^\star_{\theta,t}} \quad \textit{with} \quad P^\star_{\theta,t} = e^{-(1-\theta^\intercal \mathbf{1}) \log(1-\theta^\intercal \mathbf{1}) - \theta^\intercal \log(\theta) + \theta^\intercal \log(P_t)} \tag{21}$$

where the optimal budget fraction allocated to each product type i is

$$\varpi_{i,t} = \theta_i$$

and to cash indexed consumption is

$$1 - \varpi_t^\intercal \mathbf{1} = 1 - \theta^\intercal \mathbf{1}.$$

Proof. See Section A.8 □

As shown in Lemma 7, optimal consumption budget allocation depends only on taste parameter θ but this should not be confused with the amount of products consumed. For product i, the number of units to consume $\frac{c_t \theta_i}{P_{i,t}}$ is inversely proportional to its price $P_{i,t}$. The stochastic consumption prices vector P_t of size n_P makes the composition of the chosen bundle dynamic over time. In a dynamic setting consumption prices P_t change with time, hence the lowest attainable price of a consumption bundle $P^\star_{\theta,t}$ also varies with time.

With the solution to the consumption allocation problem Lemma 7, the lifetime decision problem (17) can then be reformulated only in terms of investment fraction π_t and consumption budget c_t decision variables with the same portfolio dynamics as in (18)

$$\begin{aligned} &J(t, W_t, X_t, P_t, Q_t) \\ &= \sup_{\pi, c \in \mathcal{A}} \mathbb{E}_t \left[\varepsilon_1 \int_t^T e^{-\int_t^s \delta_q \mathrm{d}q} u(v(c_s, P_s, \tilde{\theta}))\,\mathrm{d}s + \varepsilon_2 e^{-\int_t^T \delta_q \mathrm{d}q} u(v(W_T, P_T, \theta)) \right]. \end{aligned} \tag{22}$$

Before solving the lifecycle problem, let me introduce the mean-variance efficient portfolio, which also maximizes expected logarithmic growth. It is optimal only in some of the simplest settings, since it ignores many important problem variables. However it will help us to better understand the solution to this lifecycle investment problem.

Remark 2 (Mean-variance efficient portfolio). *Consider the self-financing $\tilde{A}_t$ portfolio with dynamics*

$$\frac{\mathrm{d}\tilde{A}_t}{\tilde{A}_t} = r_t\,\mathrm{d}t + \tilde{\pi}_t^\intercal \left(\frac{\mathrm{d}A_t}{A_t} - r_t\mathbf{1}\,\mathrm{d}t\right)$$

following the adapted investment strategy $\tilde{\pi}_t = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})$. Assuming that this strategy is square integrable, it maximizes expected logarithmic growth and mean-variance returns

$$\begin{aligned}\sup_{\tilde{\pi}_t}\mathbb{E}_t\left[\mathrm{d}\log\!\left(\tilde{A}_t\right)\right] &= \sup_{\tilde{\pi}_t}\mathbb{E}_t\left[r_t\,\mathrm{d}t + \tilde{\pi}_t^\intercal\left(\frac{\mathrm{d}A_t}{A_t} - r_t\mathbf{1}\,\mathrm{d}t\right)\right] - \frac{1}{2}\,\mathrm{d}[\tilde{\pi}_t^\intercal\Sigma_{A,t}, \tilde{\pi}_t^\intercal\Sigma_{A,t}] \\ &= \left(r_t + \frac{1}{2}\Lambda_t^\intercal\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t\right)\mathrm{d}t\end{aligned}$$

where the resulting expected return is reformulated using the no-arbitrage constraint (3)*.*

Proposed solutions are based on the dynamic programming principle. When applying this method, I implicitly assume that indirect utility is finite, once differentiable in time, twice differentiable in remaining arguments and both increasing and concave in wealth. Uniqueness of solutions is not addressed, and existence only is as much closed-form solutions from Section 5 can reach. Thus the solutions obtained through dynamic programming in this paper may be regarded formally as candidate solutions.

Admissibility should also rule out the possibility of financing consumption through debt bubbles resulting in unlimited expected utility, which can easily happen when the lack of terminal utility $\varepsilon_2 = 0$ dilutes the financial consequences of debt. These cases require of a transversality restriction analogous to that of Remark 6, which is studied at the end of this section.

Proposition 1 (Dynamic portfolio optimization). *The Hamilton-Jacobi-Bellman (HJB) equation associated to dynamic portfolio optimization problem* (22) *is given in* (A.14)*. Assume that admissibility restrictions other than adaptedness are not locally binding around optimal paths and that indirect utility is increasing and concave in wealth, giving rise to interior solutions. Optimal consumption and investment are given by*

$$c_t^\star = \varepsilon_1^{\frac{1}{\gamma}}\left(P_{\tilde{\theta},t}^\star\right)^{1-\frac{1}{\gamma}}(\partial_W J_t)^{-\frac{1}{\gamma}} \tag{23}$$

$$\pi_t^\star = \frac{\begin{aligned}\Big(&\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1}) \\ &+ \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\log(\partial_W J_t) \\ &+ \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P\log(\partial_W J_t) \\ &+ \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{QA,t}^\intercal\Sigma_{Q,t}^\intercal Q_t\partial_Q\log(\partial_W J_t)\Big)\end{aligned}}{-W_t\left(\partial_W\log(\partial_W J_t)\right)}. \tag{24}$$

where $\partial_X\log(\partial_W J_t) = \frac{\partial_{WX}J_t}{\partial_W J_t}$, indirect utility $J(t, W_t, X_t, P_t, Q_t)$ is the solution to PDE (A.15) *and $P_{\tilde{\theta},t}^\star$ is defined in* (21)*.*

Proof. See Section A.10 for the proof and the indirect utility PDE. □

The optimal investment fraction (24) from Proposition 1 has four distinct components: speculative demand to the mean-variance efficient portfolio from Remark 2 on the first line, hedging demand against changes in state X_t on the second line, hedging demand against changes in consumption prices P_t on the third line and hedging demand against changes in labor income Q_t on the fourth line. Hedging demand terms start by projecting the state X_t, price P_t or labor income Q_t risk factors against market assets A_t, resembling slope coefficients from an ordinary least squares (OLS). They hedge against state X_t, price P_t and labor income Q_t changes in proportion to the elasticity with respect to the marginal utility of saving $\partial_W J_t$. In the denominator, elasticity with respect to wealth captures the concavity of the problem. This elasticity can be interpreted as the effective relative risk aversion with respect to indirect utility, which coincides with γ in the renowned Merton (1971) problem without human capital. The investment policy π_t is expressed in relative terms as it is customary in this literature, however note that the problem is technically not well defined if $W_t = 0$ is possible. This undeterminacy disappears when the investment problem is controlled in terms of nominal exposure $W_t \pi_t$ and the relative parametrization π_t is regarded as merely illustrative.

Optimal instantaneous consumption (23) decreases in the marginal utility of saving $\partial_W J_t$. The direct effect of price $P^\star_{\tilde{\theta},t}$ on the consumption budget is positive for $\gamma > 1$ and negative for $\gamma < 1$, however the net effect is unclear at this stage since marginal utility of saving $\partial_W J_t$ is also affected by prices and can cancel out.

Proposition 2 (Solution to portfolio optimization)**.** *Indirect utility from Proposition 1 can be decomposed as*

$$J(t, W_t, X_t, P_t, Q_t) = \frac{(W_t + \Upsilon(t, X_t, Q_t))^{1-\gamma}}{1-\gamma} f(t, X_t, P_t)^\gamma \tag{25}$$

where $f(t, X_t, P_t)$ is the solution to PDE (A.19) *and human capital $\Upsilon(t, X_t, Q_t)$ coincides with Lemma 4, making the optimal policy equal to*

$$c^\star_t = \varepsilon_1^{\frac{1}{\gamma}} \left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}} \frac{W_t + \Upsilon(t, X_t, Q_t)}{f(t, X_t, P_t)} \tag{26}$$

$$\begin{aligned}
\pi^\star_t = {} & \left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1} \frac{(\mu_{A,t} - r_t \mathbf{1})}{\gamma} \frac{W_t + \Upsilon_t}{W_t} \\
& + \left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1} \Sigma_{A,t} \left(\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X \log(f_t) + \rho^\intercal_{PA,t}\Sigma^\intercal_{P,t} \operatorname{diag}(P_t)\partial_P \log(f_t)\right) \frac{W_t + \Upsilon_t}{W_t} \\
& - \left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1} \Sigma_{A,t} \frac{\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X \Upsilon_t + \rho^\intercal_{QA,t}\Sigma^\intercal_{Q,t}\Upsilon_t}{W_t}.
\end{aligned} \tag{27}$$

For this decomposition to hold, either there is no labor income $Q_t = 0$ making $\Upsilon(t, X_t, Q_t) = 0$ or markets have to be complete enough to make $\Upsilon(t, X_t, Q_t)$ replicable.

The solution to $f(t, X_t, P_t)$ corresponds to

$$f(t, X_t, P_t) = \varepsilon_2^{\frac{1}{\gamma}} \left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}} h(t, X_t; T, \theta) + \varepsilon_1^{\frac{1}{\gamma}} \left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}} \int_t^T h(t, X_t; s, \tilde{\theta}) \, \mathrm{d}s \tag{28}$$

when either there is no intermediate consumption $\varepsilon_1 = 0$, $\gamma \to 1$, markets are complete enough to make $f(t, X_t, P_t)$ replicable or $\tilde{\theta} = \theta$ with markets complete enough to make $(\partial_X f_t)^\intercal \Sigma_{X,t} \, \mathrm{d}Z_{X,t}$ replicable. $h(t, X_t; T, \theta)$ refers to the solution of PDE (A.21).

Proof. See Section A.11 □

Regarding technical aspects, PDE (A.21) for $h(t, X_t; T, \theta)$ is an instance of the semi-linear PDE for $g(t, X_t)$ described in Definition 3 and studied in Section 5. The explicit relationship between them and some additional remarks can be found at the end of Section A.11.

Optimal consumption (26) in Proposition 2 is in line with the permanent income hypothesis. Consumption is proportional to total wealth $W_t + \Upsilon_t$ and inversely proportional to the subjective consumption bundle annuity portfolio $f(t, X_t, P_t)$. Total wealth includes a human capital component $\Upsilon(t, X_t, Q_t; T_R)$ that capitalizes the labor income stream Q_t.

The optimal investment fraction (27) has a speculative component on the first line that is directly proportional to the mean-variance efficient portfolio and inversely proportional to risk aversion. Projections of subjective annuity portfolio f_t sensitivities against state X_t and prices P_t on the second line constitute hedging demand. Even though this investment policy hedges consumption prices P_t, the policy itself does not depend on price levels when $\theta = \tilde{\theta}$ as can be verified by substituting (28). Projections of human capital Υ_t sensitivities against state X_t and prices P_t on the third line are offsets compensating the builtin exposure originating from Υ_t, hence the negative sign.

Taking derivatives on both sides of (25) with respect to total wealth and rearranging terms,

$$\partial_W J_t = \partial_v u(v_t)\big|_{v_t = \frac{W_t + \Upsilon_t}{f_t}} = \varepsilon_1 \partial_c \, u(v(c_t, P_t, \theta))\big|_{c_t = \frac{W_t + \Upsilon_t}{\int_t^T h(t, X_t; s, \tilde{\theta}) \mathrm{d}s + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\gamma}} \left(\frac{P^\star_{\theta,t}}{P^\star_{\tilde{\theta},t}}\right)^{1-\frac{1}{\gamma}} h(t, X_t; T, \theta)}}$$

one can see $f(t, X_t, P_t)$ as the value of a subjective consumption bundle annuity portfolio. This annuity portfolio is risky and takes into account risk preferences, impatience, elasticity to different products, price levels, market price of risks and parameters governing those dynamics. When markets are complete, $h(t, X_t; T, \theta)$ is the price multiplier for the payoff described in Remark 3.

Remark 3. *When markets are complete enough to replicate $f(t, X_t, P_t)$ and $h(t, X_t; T, \theta)$, the solution $h(t, X_t; T, \theta)$ to PDE* (A.21) *in Proposition 2 coincides with the price multiplier $\tilde{\Omega}(t, X_t; T)$ in Lemma 3 for the claim to payoff $\hat{Q}_T$*

$$\hat{Q}_T = \hat{Q}_t \left(e^{-\int_t^T \delta_s \mathrm{d}s} \frac{\tilde{A}_T}{\tilde{A}_t} \right)^{\frac{1}{\gamma}} \left(\frac{P^\star_{\theta,T}}{P^\star_{\theta,t}} \right)^{1-\frac{1}{\gamma}}.$$

Proof. See Section A.12 □

Remark 4 (Annuities and consumption under optimal policy)**.** *In Proposition 1, wealth under the optimal policy evolves such that the number of subjective annuities at $t \in [t_0, T)$ corresponds to*

$$\frac{W_t + \Upsilon_t}{f_t} = \frac{W_{t_0} + \Upsilon_{t_0}}{f_{t_0}} \left(e^{-\int_{t_0}^t \delta_s \mathrm{d}s} \frac{\tilde{A}_t}{\tilde{A}_{t_0}} \right)^{\frac{1}{\gamma}} \exp\left(\frac{\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}}{1-\gamma} \right) \tag{29}$$

making the bundle consumption rate $v_t^\star = v(c_t^\star, P_t, \tilde{\theta})$ equal to

$$v_t^\star = v_{t_0}^\star \left(e^{-\int_{t_0}^t \delta_s \mathrm{d}s} \frac{P^\star_{\tilde{\theta},t_0}}{P^\star_{\tilde{\theta},t}} \frac{\tilde{A}_t}{\tilde{A}_{t_0}} \right)^{\frac{1}{\gamma}} \exp\left(\frac{\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}}{1-\gamma} \right).$$

where and $\tilde{A}_t$ is the mean-variance efficient portfolio from Remark 2, the exponential term on the right tends to $\frac{\tilde{\zeta}_t}{\tilde{\zeta}_{t_0}}\frac{\zeta_{t_0}}{\zeta_t}$ as $\gamma \to 1$ and the process pair $\zeta_t, \tilde{\zeta}_t$ captures the degree of market incompleteness for $f(t, X_t, P_t)$

$$\begin{aligned}\frac{\mathrm{d}\zeta_t}{\zeta_t} =&(\partial_X \log(f_t))^\intercal \Sigma_{X,t}\,\mathrm{d}Z_{X,t} + (\partial_P \log(f_t))^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\,\mathrm{d}Z_{P,t}\\ \frac{\mathrm{d}\tilde{\zeta}_t}{\tilde{\zeta}_t} =& \left((\partial_X \log(f_t))^\intercal \Sigma_{X,t}\rho_{XA,t} + (\partial_P \log(f_t))^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\right)\mathcal{P}_{\Sigma_{A,t}^\intercal}\,\mathrm{d}Z_{A,t}.\end{aligned} \tag{30}$$

Proof. See Section A.13. □

The bundle consumption rate formula in Remark 4 shows explicitly how it decreases with impatience, decreases with prices and increases with cumulative returns of the mean-variance efficient portfolio in a proportion inverse to risk aversion γ, up to a process pair capturing market incompleteness. The number of lifetime annuities (29), that is the ratio of total wealth to the annuity portfolio value, decreases over time with the impatience rate and increases with the cumulative returns of the mean-variance efficient portfolio in a proportion inverse to risk aversion γ, up to the market incompleteness process pair. In case of extreme risk aversion and complete markets, the bundle consumption rate and the number of lifetime annuities become constant (Remark 5).

Remark 5 (Extreme risk aversion). *Extreme risk aversion $\gamma \to \infty$ when markets are complete enough to make $f(t, X_t, P_t)$ replicable makes the bundle consumption rate $v(c_t^\star, P_t, \tilde{\theta})$ and the number of lifetime annuities in Remark 4 constant*

$$v(c_t^\star, P_t, \tilde{\theta}) = \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)} = \frac{W_{t_0} + \Upsilon(t_0, X_{t_0}, Q_{t_0}; T_R)}{f(t_0, X_{t_0}, P_{t_0})}.$$

Similar to Merton (1971), the lifecycle objective with a random terminal time is under certain conditions equivalent to an infinite horizon objective with an increased impatience rate as explained in Remark 6.

Remark 6 (Random terminal time). *Suppose that there is only intermediate consumption $\varepsilon_1 = 1, \varepsilon_2 = 0$ and that instead of terminal time T there is a random stopping time τ given by the first arrival of a Poisson process with state dependent hazard rate $\lambda_{T,t} := \lambda_T(t, X_t)$. Assume that each possible hazard rate path $\lambda_{T,[t,\infty]}$ satisfies*

$$\lambda_{T,s} \geq 0 \quad \forall s \geq t \quad \text{and} \quad \int_t^\infty \lambda_{T,s}\,\mathrm{d}s = \infty. \tag{31}$$

and also that the intermediate utility integral below converges to a finite value for the optimal policy under each possible hazard rate path $\lambda_{T,[t,\infty]}$

$$\int_t^\infty \mathbb{E}\left[e^{-\int_t^s \delta_q \mathrm{d}q} u(v(c_s, P_s, \tilde{\theta}))|\mathcal{F}_t, \lambda_{T,[t,\infty]}\right]\mathrm{d}s\,. \tag{32}$$

Then the expectation over discounted intermediate utility for the optimal policy conditional on the individual having survived up to time t is equivalent to

$$\mathbb{E}_t\left[\int_t^\tau e^{-\int_t^s \delta_q \mathrm{d}q} u(v(c_s, P_s, \tilde{\theta}))\,\mathrm{d}s\right] = \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s (\lambda_{T,q}+\delta_q)\mathrm{d}q} u(v(c_s, P_s, \tilde{\theta}))\,\mathrm{d}s\right].$$

Proof. See Section A.9 □

The most important condition for Remark 6 is (32). It can be related to the dilution of financial consequences upon sudden termination or in infinite horizon settings, because they could induce some pathological behavior and make the problem not well defined. For the stated equivalence to hold, expected discounted utility in (32) should remain finite for the optimal policy subject to admissibility constraints under each possible hazard rate path. This condition is still compatible with lax admissibility constraints that allow individuals to leave a negative wealth balance with positive probability upon death, but the equivalence holds only when individuals cannot exploit this peculiarity to obtain unlimited expected utility.

When considering a random terminal stopping time τ with labor income $Q_t \neq 0$, wealth dynamics (18) should condition receiving the labor income stream Q_t on not having reached retirement T_R and not having reached terminal time τ, replacing the previous labor income term with $Q_t \mathbb{1}_{t \leq (T_R \wedge \tau)}$. This means that Proposition 1 and Proposition 2 do not apply to the problem transformed in Remark 6 when $Q_t \neq 0$.

3 Welfare changes

To measure welfare changes in this framework, I propose to use an equivalent initial wealth (EW) concept in line with Kraft and Munk (2011) instead of certainty equivalent terms. Using certainty equivalents is customary in the financial literature but it poses some problems in this specific setting. Asking for a certain amount of money as compensation to remove investment risk leaves unaddressed the issue of consumption price risk and the utility derived from consumption. Consumption prices can change from the beginning to the end of the investment horizon. If markets are incomplete and prices cannot be completely hedged, terminal consumption derived from a certain compensation amount is uncertain making certainty equivalents undefined. The equivalent initial wealth concept imposes weaker restrictions and asks for the certain compensation amount needed to replace one investment strategy with another while providing an equivalent level of expected utility.

Definition 2 (Equivalent initial wealth)**.** *The equivalent initial wealth* EW *is the minimum amount of initial wealth $\tilde{W}$ at time t needed to compensate the substitution of the lottery associated to investment strategy π over initial wealth W_t by the forced choice of investment strategy $\bar{\pi}$ for an individual with expected utility given by $U(t, W_t; \pi)$ given some adapted vector process H_t.*

$$\begin{aligned} \mathrm{EW}(t, W_t, H_t; \pi, \bar{\pi}) = \min_{\tilde{W}} \; & \tilde{W} \\ s.t. \; & U(t, \tilde{W}, H_t; \bar{\pi}) \geq U(t, W_t, H_t; \pi). \end{aligned} \tag{33}$$

Lemma 8 (Equivalent initial wealth)**.** *Suppose that the expected utility $U(t, W_t, H_t; \pi)$ is, for any π, continuous and strictly monotonically increasing in initial wealth W_t, and the range of the function does not depend on π. Then the equivalent initial wealth in the sense of Definition 2 corresponds to*

$$\mathrm{EW}(t, W_t, H_t; \pi, \bar{\pi}) = \left(U(t, \cdot, H_t; \bar{\pi})\right)^{-1} \left(U(t, W_t, H_t; \pi)\right). \tag{34}$$

Proof. See Section A.16 □

Lemma 9 (Expected utility without consumption or human capital)**.** *Suppose that* $H_t = (X_t^\intercal, P_t^\intercal)^\intercal$ *and that an individual applies investment strategy* π*, which is not necessarily optimal and depends only on time and state* X_t*. Assuming that there is no labor income* $Q_t = 0$ *or consumption* $\varepsilon_1 = 0, \varepsilon_2 = 1$*, expected utility is*

$$U(t, W_t, X_t, P_t; \pi) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t, X_t, P_t; \pi)^\gamma$$

where

$$f(t, X_t, P_t; \pi) = \left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}} h(t, X_t; T, \theta, \pi)$$

and $h(t, X_t; T, \theta, \pi)$ *solves PDE* (A.34)*.*

Proof. See Section A.17 □

PDE (A.34) for $h(t, X_t; T, \theta, \pi)$ is an instance of the semi-linear PDE for $g(t, X_t)$ described in Definition 3 and studied in Section 5. The explicit relationship between them and some additional remarks can be found at the end of Section A.17. Substituting $\pi_t^\star$ from (27) into PDE (A.34) coincides with PDE (A.21) for $h(t, X_t; T, \theta)$ from Section 2 after applying previously invoked assumptions. When evaluating time-dependent investment strategies π and under appropriate conditions, Corollary 3 on forthcoming Section 5 will be useful to isolate and solve time dependent terms separately on this PDE.

Corollary 1 (Equivalent initial wealth with incomplete dynamic markets)**.** *Assuming that there is no labor income* $Q_t = 0$ *or consumption* $\varepsilon_1 = 0, \varepsilon_2 = 1$*, the equivalent initial wealth* EW *is*

$$\mathrm{EW}(t, W_t, X_t; \pi, \bar{\pi}) = W_t \left(\frac{h(t, X_t; T, \theta, \pi)}{h(t, X_t; T, \theta, \bar{\pi})}\right)^{\frac{\gamma}{1-\gamma}}.$$

Proof. Straightforward application of Lemma 8 to expected utility from Lemma 9. □

Measuring welfare changes in terms of equivalent wealth growth rates as proposed below facilitates comparability. It captures how the compensation multiplier grows with the time horizon. This unit has the same interpretation for any time horizon and is comparable across different wealth levels.

$$\text{Equivalent wealth growth} = \frac{1}{T-t} \log\left(\frac{\mathrm{EW}(t, W_t, X_t; \pi, \bar{\pi})}{W_t}\right)$$

Assuming that human capital markets are complete enough, this welfare measure generalizes to total wealth $W_t' = W_t + \Upsilon_t$ when investment strategies π and $\bar{\pi}$ operate over total wealth or are appropriately converted to do so.

4 Stationarity

In this section I argue that lifecycle paths of specific individuals or heterogeneous populations are comparable across different generations under certain stationary conditions. Individuals, like their ancestors, enter the workforce with little financial wealth in comparison to salary and consumption price levels. They work until retirement receiving some exogenous income and accumulate enough savings to maintain their consumption capacity during retirement. During their lifetime, they also take advantage of investment opportunities.

To make this comparison possible, I argue in favor of normalizing the consumption bundle by the purchasing power of the reference or "typical" income at each point in time. By construction, this normalization should not meaningfully affect consumption price categories that increase proportionally to salaries, including labor intensive services and scarce goods with prices determined by aggregate purchasing power when this is related to reference income levels. Housing, education and health care are good examples and they constitute large shares of household consumption budgets. For remaining consumption categories, this argument implies that people derive utility in comparison to the quantity that the reference income can afford. Some reasons to "keep up with the Joneses" may originate from a desire to socialize on equal terms, or from a desire to provide for their families and children according to society norms. In this setting individuals feel entitled to contemporaneous productivity enhancements and suffer disutility if they are left behind in terms of technological progress.

Suppose that there is a common factor Y_t shared among optimal consumption bundle prices, the labor income process, and initial wealth

$$P^\star_{\theta,t} = Y_t e^{\nu_{P^\star_\theta,t}} \qquad P^\star_{\tilde{\theta},t} = Y_t e^{\nu_{P^\star_{\tilde{\theta}},t}} \qquad Q_t = Y_t e^{\nu_{Q,t}} \qquad W_{t_0} = Y_{t_0} e^{\nu_{W_{t_0}}}$$

where log wedges $\nu_{P^\star_\theta,t} := \nu_Q(X_t), \nu_{P^\star_{\tilde{\theta}},t} := \nu_{P^\star_{\tilde{\theta}}}(X_t)$ may only depend on state X_t, $\nu_{Q,t} := \nu_Q(t - t_0, X_t)$ may only depend on age $t - t_0$ and state X_t and $\nu_{W_{t_0}} := \nu_{W_{t_0}}(X_{t_0})$ may only depend on initial state X_{t_0}. The strictly positive reference scalar process Y_t can be interpreted as the average income process in a certain region and serves as cointegration factor for life-cycle processes. It is modelled to have state dependent drift $\mu_{Y,t} := \mu_Y(X_t)$ and diffusion matrix $\Sigma_{Y,t} := \Sigma_Y(X_t)$ driven by a standard Brownian motion vector $Z_{Y,t}$ with instantaneous correlations $\rho_{YA,t} := \rho_{YA}(X_t)$ and $\rho_{XY,t} := \rho_{XY}(X_t)$ to previously defined Brownian motion vectors

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_{Y,t}\,\mathrm{d}t + \Sigma_{Y,t}\,\mathrm{d}Z_{Y,t}\,.$$

Further assume that $r_t = r(X_t)$, $\mu_{X,t} = \mu_X(X_t)$, $\Sigma_{X,t} = \Sigma_X(X_t)$, $\mu_{A,t} = \mu_A(X_t)$, $\Sigma_{A,t} = \Sigma_A(X_t)$ and $\rho_{XA,t} = \rho_{XA}(X_t)$ are purely dependent on state X_t while $\delta_t = \delta(t - t_0, X_t)$ may only depend on age $t - t_0$ and state X_t. Then under the following reparametrization

$$\tilde{c}_t = \frac{c_t}{Y_t} \qquad \tilde{W}_t = \frac{W_t}{Y_t} \qquad \tau_{T,t} = (T - t_0) - (t - t_0) \qquad \tau_{R,t} = (T_R - t_0) - (t - t_0),$$

the portfolio optimization problem (22) becomes

$$\begin{aligned}
&J(t - t_0, \tilde{W}_t, X_t) \\
&= \sup_{\pi,\tilde{c} \in \tilde{\mathcal{A}}} \mathbb{E}_t \left[\varepsilon_1 \int_0^{\tau_{T,t}} e^{-\int_0^s \delta_{t+q} \mathrm{d}q} u(\tilde{c}_{t+s} e^{-\nu_{P^\star_{\tilde{\theta}},t+s}})\,\mathrm{d}s + \varepsilon_2 e^{-\int_0^{\tau_{T,t}} \delta_{t+q}\mathrm{d}q} u(\tilde{W}_{t+\tau_{T,t}} e^{-\nu_{P^\star_{\theta},t+\tau_{T,t}}}) \right] \\
&\quad \text{s.t. } \mathrm{d}\tilde{W}_t = \left(e^{\nu_{Q,t}} \mathbb{1}_{\tau_{R,t} \geq 0} - \tilde{c}_t\right) \mathrm{d}t \\
&\qquad\qquad + \tilde{W}_t \left(\left(r_t + \Sigma_{Y,t}\Sigma_{Y,t}^\intercal\right) \mathrm{d}t - \frac{\mathrm{d}Y_t}{Y_t} + \pi_t^\intercal \left(\frac{\mathrm{d}A_t}{A_t} - \left(r_t \mathbf{1} + \Sigma_{A,t}\rho_{YA,t}^\intercal \Sigma_{Y,t}^\intercal\right) \mathrm{d}t \right) \right)
\end{aligned} \tag{35}$$

where admissibility $\mathcal{A}$ is assumed to have an equivalent formulation $\tilde{\mathcal{A}}$ in terms of $\tilde{W}_t$, $\tilde{c}_t$, X_t, π_t and $t - t_0$ without needing direct references to W_t, c_t or t.

Theorem 1 (Stationary lifecycle paths)**.** *Assuming that state, asset return increment and reference process increment paths are stationary, i.e. the unconditional distribution over any interval* $[t_1, t_n]$ *is invariant for any time shift* h

$$\left\{X_{[t_1,t_n]}, \Delta Z_{A,[t_1,t_n]}, \Delta Z_{Y,[t_1,t_n]}\right\} \overset{d}{\sim} \left\{X_{[t_1+h,t_n+h]}, \Delta Z_{A,[t_1+h,t_n+h]}, \Delta Z_{Y,[t_1+h,t_n+h]}\right\}, \tag{36}$$

then the paths of lifecycle processes below associated to problem (35) *are also stationary*

$$\tilde{W}_t, \pi_t^\star, \tilde{W}_t\pi_t^\star, \tilde{c}_t^\star, \frac{c_t^\star}{P_{\tilde{\theta},t}^\star}, J_t, \frac{Q_t}{Y_t}, \frac{P_{\theta,t}^\star}{Y_t}, \frac{P_{\tilde{\theta},t}^\star}{Y_t}. \tag{37}$$

This holds also under relaxed assumptions allowing parameters τ_{T,t_0}, τ_{R,t_0}, γ, θ, $\tilde{\theta}$, ε_1, ε_2 *to depend on* X_{t_0}, *or when modelling* $\tau_{T,t}$, $\tau_{R,t}$ *as random stopping times given by the first arrival since* t_0 *of their respective Poisson process with non-negative hazard rate process* $\lambda_{T,t} \coloneqq \lambda_T(t - t_0, X_t)$ *for* $\tau_{T,t}$ *and* $\lambda_{R,t} \coloneqq \lambda_R(t - t_0, X_t)$ *for* $\tau_{R,t}$ *dependent on age* $t - t_0$ *and state* X_t.

Proof. See Section A.14 □

For these investors the reference process Y_t and cointegrated processes Q_t, $P_{\theta,t}^\star$, $P_{\tilde{\theta},t}^\star$ become natural numéraires. Lifecycle paths can be conviently compared when expressing wealth in years of reference income saved $\tilde{W}_t$ and consumption rates in proportion to the reference income $\tilde{c}_t^\star$.

Lifecycle paths even become deterministic under Corollary 2 when risk aversion is extreme, product elasticities of intermediate and terminal consumption coincide $\tilde{\theta} = \theta$, markets are complete and there is no state process X_t.

Corollary 2 (Deterministic lifecycle processes under extreme risk aversion)**.** *Assuming there is no state process* X_t, *ratios* $\frac{P_{\theta,t}^\star}{Y_t}, \frac{P_{\tilde{\theta},t}^\star}{Y_t}$ *become constant and additionally, under Remark 5 conditions of the extreme risk aversion and complete markets plus the condition* $\tilde{\theta} = \theta$, *problem* (35) *yields deterministic processes*

$$\tilde{W}_t, J_t, \frac{Q_t}{Y_t}, \frac{f_t}{Y_t}, \frac{\Upsilon_t}{Y_t}.$$

with constant $\tilde{c}_t^\star, \frac{c_t^\star}{P_{\tilde{\theta},t}^\star}$ *and constant investment fraction* $\pi_t^\star$ *hedging completely against* Y_t

$$\pi_t^\star = \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{YA}^\intercal\Sigma_Y^\intercal.$$

Proof. See Section A.15. □

5 Solving semi-linear PDEs

As seen in previous sections, the dynamics of this model depend on a particular type of semi-linear PDEs defined in Definition 3. The methodology used to reduce this type of semi-linear PDEs into a system of Riccati ODEs and to solve them is based on Kim and Omberg (1996), Wachter (2002) and Liu (2007) with some additional extensions. Solving these PDEs is based on repeated separation of the main semi-linear PDE into two semi-linear PDEs of the same type with reduced state spaces (Lemma 10). If the PDE admits a log-quadratic representation, it can be reduced to a system of coupled Riccati ordinary differential equations (ODEs) as explained in Lemma 11 with closed form solutions detailed in Section A.22. When imposing quadratic and affine structures, the Riccati ODEs include as special cases those of zero-coupon bond prices with quadratic term structures from Ahn et al. (2002). Existence of solutions is proven only as far as those closed form solutions can reach, while uniqueness is not addressed. In general a solution may not exist, or exist only under some conditions, like a time horizon smaller than some threshold. That said, these kind of PDEs are well known in the finance literature.

Definition 3 (Semi-linear PDE). *The partial differential equation (PDE) for $g(t, X_t)$ with boundary condition $g(T, X_t)$ and dynamic coefficients $R_t := R(t, X_t)$ as a scalar, $B_t := B(t, X_t)$ of size n_X, $C_t := C(t, X_t)$ symmetric of size $n_X \times n_X$ and $D_t := D(t, X_t)$ symmetric of size $n_X \times n_X$*

$$0 =\partial_t g_t + g_t R_t + (\partial_X g_t)^\intercal B_t + \frac{1}{2}\frac{(\partial_X g_t)^\intercal (C_t - D_t)\, \partial_X g_t}{g(t, X_t)} + \frac{1}{2}\operatorname{tr}\left(D_t \partial_{XX^\intercal} g_t\right). \tag{38}$$

My main contribution compared to Liu (2007) is to separate these semi-linear PDEs allowing for the off-diagonal terms in C_t, D_t and asymmetries explained in Lemma 10, which also expands the range of analytic solutions to the Riccati ODEs in Lemma 11. These enhancements allow under certain conditions to solve the pricing PDE (8) for $\tilde{\Omega}(t, X_t; T)$ (Lemma 3), the stochastic control PDE (A.21) for $h(t, X_t; T, \theta)$ (Proposition 2) and the expected utility PDE (A.34) for $h(t, X_t; T, \theta, \pi)$ (Lemma 9) in a setting where state components are related to each other and have non-diagonal covariance matrices. Note that parameters C_t, D_t comprise the state covariance matrix $\Sigma_{X,t}\Sigma_{X,t}^\intercal$ and its market projection $\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal$ in PDE (A.21) for $h(t, X_t; T, \theta)$. Also Corollary 3 allows to isolate time dependents easily, which is very useful to evaluate time dependent investment strategies in PDE (A.34) for $h(t, X_t; T, \theta, \pi)$.

Lemma 10 (Semi-linear PDE separation). *Suppose that in PDE* (38) *described in Definition 3 there are two state subsets $X_t = (X_{1,t}^\intercal, X_{2,t}^\intercal)^\intercal$ such that the boundary condition admits the following split product structure $g(T, X_t) = g_1(T, X_{1,t}) g_2(T, X_{2,t})$ and that dynamic coefficients can be decomposed as follows*

$$R_t = R_{1,t} + R_{2,t} \quad B_t = \begin{pmatrix} B_{1,t} \\ B_{2,t} + \hat{B}_{1,t} \end{pmatrix} \quad C_t = \begin{pmatrix} C_{1,t} & \tilde{C}_{1,t} \\ \tilde{C}_{1,t}^\intercal & C_{2,t} + \hat{C}_{1,t} \end{pmatrix} \quad D_t = \begin{pmatrix} D_{1,t} & D_{\star,t} \\ D_{\star,t}^\intercal & D_{2,t} + \hat{D}_{1,t} \end{pmatrix} \tag{39}$$

where $D_{\star,t} := D_\star(t, X_t)$ is unrestricted but $R_{i,t} := R_i(t, X_{i,t})$, $B_{i,t} := B_i(t, X_{i,t})$, $\hat{B}_{i,t} := \hat{B}_i(t, X_{i,t})$, $C_{i,t} := C_i(t, X_{i,t})$, $\hat{C}_{i,t} := \hat{C}_i(t, X_{i,t})$, $\tilde{C}_{i,t} := \tilde{C}_i(t, X_{i,t})$, $D_{i,t} := D_i(t, X_{i,t})$ and $\hat{D}_{i,t} := \hat{D}_i(t, X_{i,t})$ for $i \in \{1, 2\}$ may depend only on their state indicator $X_{i,t}$ or time. When satisfying

1. *Conditional log-linearity. Either $\partial_{X_2} \log g_{2,t}$ does not depend on $X_{2,t}$ or $\hat{B}_{1,t}, \hat{C}_{1,t}, \tilde{C}_{1,t}$ are zero.*
2. *Conditional log-quadraticity. Either $\partial_{X_2 X_2^\intercal} \log g_{2,t}$ does not depend on $X_{2,t}$ or $\hat{D}_{1,t}$ is zero.*

then the solution to PDE (38) *can be decomposed into*

$$g(t, X) =g_1(t, X_{1,t}) g_2(t, X_{2,t}) \tag{40}$$

where $g_2(t, X_{2,t})$ solves the next PDE with boundary condition $g_2(T, X_{2,t})$

$$0 = \partial_t g_{2,t} + g_{2,t} R_{2,t} + (\partial_{X_2} g_{2,t})^\intercal B_{2,t} + \frac{1}{2} \frac{(\partial_{X_2} g_{2,t})^\intercal (C_{2,t} - D_{2,t}) \partial_{X_2} g_{2,t}}{g_2(t, X_{2,t})} + \frac{1}{2} \operatorname{tr}\left(D_{2,t} \partial_{X_2 X_2^\intercal} g_{2,t}\right) \tag{41}$$

and $g_1(t, X_{1,t})$ solves this other PDE with boundary condition $g_1(T, X_{1,t})$

$$\begin{aligned} 0 =& \partial_t g_{1,t} + g_{1,t} \Big(R_{1,t} + (\partial_{X_2} \log g_{2,t})^\intercal \hat{B}_{1,t} + \frac{1}{2} (\partial_{X_2} \log g_{2,t})^\intercal \hat{C}_{1,t} \partial_{X_2} \log g_{2,t} \\ &+ \frac{1}{2} \operatorname{tr}\left(\hat{D}_{1,t} \partial_{X_2 X_2^\intercal} \log g_{2,t}\right) \Big) \\ &+ (\partial_{X_1} g_{1,t})^\intercal \left(B_{1,t} + \tilde{C}_{1,t} \partial_{X_2} \log g_{2,t}\right) + \frac{1}{2} \frac{(\partial_{X_1} g_{1,t})^\intercal (C_{1,t} - D_{1,t}) \partial_{X_1} g_{1,t}}{g_1(t, X_{1,t})} + \frac{1}{2} \operatorname{tr}\left(D_{1,t} \partial_{X_1 X_1^\intercal} g_{1,t}\right). \end{aligned} \tag{42}$$

Note that PDEs (41) (42) are in turn semi-linear PDEs of the same type as (38) described in Definition 3 and the separation procedure may be applied repeatedly as long as the aforementioned conditions hold.

Proof. See Section A.18. □

Corollary 3 (Time separation). *Invoking Lemma 10 with an empty state partition $X_{1,t}$ reduces (42) to an ODE for $g_1(t)$ with solution*

$$\begin{aligned} g_1(t) =& g_1(T) \exp\Bigg(\int_t^T \Big(R_{1,s} + (\partial_{X_2} \log g_{2,s})^\intercal \hat{B}_{1,s} + \frac{1}{2} (\partial_{X_2} \log g_{2,s})^\intercal \hat{C}_{1,s} \partial_{X_2} \log g_{2,s} \\ &+ \frac{1}{2} \operatorname{tr}\left(\hat{D}_{1,s} \partial_{X_2 X_2^\intercal} \log g_{2,s}\right) \Big) \, \mathrm{d}s \Bigg). \end{aligned}$$

Proof. See Section A.19 □

Lemma 11 (Semi-linear PDE to Riccati ODEs). *The semi-linear PDE from Definition 3 can be reduced to a system of Riccati ordinary differential equations (ODEs) under some restrictions as detailed in Section A.21, and when these Riccati equations satisfy a diagonal strucutre, they can be solved in closed form as detailed in Section A.22.*

6 Applications

Below I give some examples of models with analytic PDE solutions. This paper retains full compatibility with Liu (2007) quadratic and affine models, including closed form solutions cover the case of stochastic price of risk and the stochastic volatility model with a market price of risk proportional to volatility. These models yield quadratic building blocks for $h(t, X_t; T, \theta)$ described in Section A.11 in relation to Proposition 2.

Housing and income cointegration In García (2026) I model how lifecycle investing is affected by cointegration between housing rent and income processes. The state process X_t has Ornstein-Uhlenbeck dynamics and comprises a mean reverting housing cycle component embedded in rent prices P_t, income Q_t and housing prices $\Upsilon_{H,t}$

$$\mu_{X,t} = \alpha_X - \beta_X \odot X_t.$$

Assuming that other parameters are constant, the following building blocks are linear in X_t

$$\begin{aligned}
\mu_{X,t} &= \alpha_X - \beta_X \odot X_t \\
\mu_{P,t} &= \alpha_P + \beta_P X_t \\
\mu_{Q,t} &= \alpha_Q + \beta_Q X_t,
\end{aligned}$$

and remaining building blocks are constant. All building blocks containing covariance and correlation terms, e.g. Σ_X, Σ_A or ρ_{XA}, can be have off-diagonal elements. House prices $\Upsilon_{H,t}$ understood as the market value of future rents P_t depend on the housing cycle X_t, however the return dynamics for investing in housing A_t do not depend on state X_t since Ornstein-Uhlenbeck processes have constant volatility. This model would remain equally tractable if extended with a stochastic short-term risk-free rate

$$r_t = \alpha_r + \beta_r X_t.$$

Stochastic market price of risk Similar to Kim and Omberg (1996), Wachter (2002) and Liu (2007), there is a stochastic market price of risk vector

$$\Lambda_t = \alpha_\Lambda + \beta_\Lambda X_t$$

where state X_t is an Ornstein-Uhlenbeck process with mean reverting dynamics

$$\mu_{X,t} = \alpha_X - \beta_X X_t.$$

Assuming that other parameters are constant, the following building blocks are linear in X_t

$$\begin{aligned}
\mu_{X,t} &= \alpha_X - \beta_X X_t \\
\Sigma_P \rho_{PA} \mathcal{P}_{\Sigma_A^\intercal} \Lambda_t &= \Sigma_P \rho_{PA} \mathcal{P}_{\Sigma_A^\intercal} \left(\alpha_\Lambda + \beta_\Lambda X_t\right) \\
\Sigma_X \rho_{XA} \mathcal{P}_{\Sigma_A^\intercal} \Lambda_t &= \Sigma_X \rho_{XA} \mathcal{P}_{\Sigma_A^\intercal} \left(\alpha_\Lambda + \beta_\Lambda X_t\right),
\end{aligned}$$

this one is quadratic

$$\Lambda_t^\intercal \mathcal{P}_{\Sigma_A^\intercal} \Lambda_t = \alpha_\Lambda^\intercal \mathcal{P}_{\Sigma_A^\intercal} \alpha_\Lambda + 2\alpha_\Lambda^\intercal \mathcal{P}_{\Sigma_A^\intercal} \beta_\Lambda X_t + X_t^\intercal \beta_\Lambda^\intercal \mathcal{P}_{\Sigma_A^\intercal} \beta_\Lambda X_t,$$

and remaining building blocks are constant. Closed form solutions from Section A.22 become applicable when Σ_X,ρ_{XA},Σ_A,β_Λ,β_X are diagonal.

Heston (1993) stochastic volatility An interesting example in Liu (2007) features stochastic volatility with market price of risk proportional to volatility. I extend this model to the case where both asset returns and consumption prices share a common volatility component with possibly different loadings. This could capture for instance a shared volatility component in both housing rents and housing investment returns. However the case of Liu (2007) with stochastic volatility only in asset returns is still solvable in closed form when returns are independent from consumption price changes, or consumption prices are constant. The diffusion matrix for risky investment returns and the market price of risk are

$$\begin{aligned}
\Sigma_{A,t} =& \operatorname{diag}\left(\sqrt{X_t}\right) \\
\Lambda_t =& \Sigma_{A,t}^{\intercal}\bar{\Lambda}.
\end{aligned}$$

The state X_t process captures the variance of asset returns and it follows a mean reverting Cox-Ingersoll-Ross process with

$$\begin{aligned}
\mu_{X,t} =& \alpha_X - \beta_X \odot X_t \\
\Sigma_{X,t} =& \operatorname{diag}\left(L_X \odot \sqrt{X_t}\right).
\end{aligned}$$

The Feller condition $2\alpha_X \geq L_X^2$ guarantees that all components are strictly positive. Assuming that ρ_{PA}, ρ_{XA}, ρ_{XP} are constant diagonal matrices and that volatility of consumption prices corresponds to

$$\Sigma_{P,t} = \operatorname{diag}\left(L_P \odot \sqrt{X_t}\right),$$

the following building blocks become linear in state X_t

$$\begin{aligned}
\mu_{X,t} =& \alpha_X - \beta_X \odot X_t \\
\Lambda_t^{\intercal}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\Lambda_t =& (\bar{\Lambda}^2)^{\intercal} X_t \\
\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\Lambda_t =& \operatorname{diag}(\rho_{PA}) \odot L_P \odot \bar{\Lambda} \odot X_t \\
\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\rho_{PA,t}^{\intercal}\Sigma_{P,t}^{\intercal} =& \rho_{PA}^2 \operatorname{diag}\left(L_P^2 \odot X_t\right) \\
\Sigma_{P,t}\Sigma_{P,t}^{\intercal} =& \operatorname{diag}\left(L_P^2 \odot X_t\right) \\
\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\Lambda_t =& \operatorname{diag}(\rho_{XA}) \odot L_X \odot \bar{\Lambda} \odot X_t \\
\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^{\intercal} =& \rho_{XP} \operatorname{diag}(L_X \odot L_P \odot X_t) \\
\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\rho_{PA,t}^{\intercal}\Sigma_{P,t}^{\intercal} =& \rho_{XA}\rho_{PA} \operatorname{diag}(L_X \odot L_P \odot X_t) \\
\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal} =& \rho_{XA}^2 \operatorname{diag}\left(L_X^2 \odot X_t\right) \\
\Sigma_{X,t}\Sigma_{X,t}^{\intercal} =& \operatorname{diag}\left(L_X^2 \odot X_t\right)
\end{aligned}$$

and remaining building blocks are constant. Imposing a diagonal structure on a linear model like this may seem overly restrictive since Lemma 10 can deal with off-diagonal elements in semi-linear PDEs (Definition 3). The main reason is to prevent non-linear $\sqrt{X_t}$ terms from breaking the linearity of building blocks above, and it also allows to conveniently extend the scalar Feller condition in this vectorized setting by keeping the variance processes isolated.

7 Conclusion

This lifecycle model solves the consumption and investment problems in a setting in which stochastic investment returns, stochastic income and stochastic consumption prices share a common state process. As in Liu (2007), exact solutions are available when stochastic processes follow certain quadratic cases.

When investment instruments like stock shares, bonds or houses are understood as claims to future payoffs or payoff streams, the asset prices can be decomposed into the current payoff process value and a price multiplier. Under some state process stationarity conditions, financial indicators based on price multipliers also become stationary.

The optimal lifecycle policy balances exposure to risky investment assets, consumption price risk and income risk such that the resulting bundle consumption rate is exposed to the mean-variance efficient portfolio and consumption price risk in inverse proportion to risk aversion.

Under some stationarity conditions, lifecycle paths of specific individuals or heterogeneous populations become comparable across different generations. The reference income process becomes a natural numeraire for individuals whose consumption is exposed mainly to labor price risk or wanting to keep up with the Joneses. One may then prefer to measure savings as years of average income saved and consumption rates in proportion to average income.

A Appendix

A.1 Notation

$\square \odot \square$ Hadamard product, element-wise matrix multiplication

I identity matrix of appropriate size according to context

$\mathbf{0}$ vector of zeros of appropriate size according to context

$\mathbf{1}$ vector of ones of appropriate size according to context

$\mathbb{1}_{\square}$ indicator function for predicate, applied elementwise if predicate is vector or matrix-valued

$e^{\square}, \exp(\square)$ exponential function, applied element-wise to vectors

$\operatorname{diag}(\square)$ creates a diagonal matrix if the argument is a vector, or extracts the diagonal vector from a square matrix if the argument is a matrix

$\operatorname{tr}(\square)$ matrix trace

$(\square)^{-1}$ matrix inverse or reciprocal when applied to scalars

$f(x, \cdot, z)^{-1}(m)$ function inverse for argument at position $\cdot$ corresponding to output m

$\partial_{\square} f(t, X_t)$ partial derivative of function f with respect to specified arguments in order. Repeated arguments indicate higher order derivatives. It may also be used on expressions that are treated as function compositions, e.g. $\partial_{\square} \log(f(t, X_t))$.

$\mathrm{d}\square$ differential operator and it refers to the Itô differential when applied stochastic expressions

$\frac{\mathrm{d}A_t}{A_t}$ on a vector process A_t is shorthand notation for $\operatorname{diag}(A_t)^{-1}\,\mathrm{d}A_t$

$A_{[t,T]}$ path of A_t from time t to time T

$\Delta A_{[t,T]}$ path of increments $A_s - A_t$ for the interval $s \in [t, T]$

$\square \overset{d}{\sim} \square$ expressions have the same distribution

$\mathbb{E}_t[\Box]$ expectation conditional on the filtration up to time t, i.e. $\mathbb{E}[\Box|\mathcal{F}_t]$

Function expressions may use semi-colon ; after function arguments to specify parameters, like $\Omega(t, X_t, Q_t; T)$ referring to PDE solution from Lemma 3. They may also use the equal sign = to assign expressions to parameters by name, like $g\left(t, X_t; R_t = \Box, B_t = \Box, C_t = \Box, D_t = \Box\right)$ referring to PDE solution from Definition 3.

Some proofs use tensor notation, which is briefly explained in Section A.20.

A.2 Proof of span and replicability quadratic variation, Lemma 2

The "if" direction follows directly from applying Definition 1. To prove the "only if" direction for the first statement, suppose that φ_t is not completely spanned by asset risk factors $Z_{A,t}$

$$\Sigma_{\varphi,t}\left(\mathrm{d}Z_{\varphi,t} - \rho_{\varphi A,t}\,\mathrm{d}Z_{A,t}\right) \neq 0.$$

This process can be separated into its $Z_{A,t}$ projection corresponding to the right hand side of (4) and its remainder component orthogonal below. Because of this orthogonality, the quadratic variation of φ_t is the sum of the quadratic variation of its $Z_{A,t}$ projection plus that of the remainder

$$\mathrm{d}[\varphi_t, \varphi_t] = \underbrace{\Sigma_{\varphi,t}\rho_{\varphi A,t}\rho_{\varphi A,t}^{\intercal}\Sigma_{\varphi,t}^{\intercal}\,\mathrm{d}t}_{Z_{A,t}\text{ projection}} + \underbrace{\Sigma_{\varphi,t}\left(I - \rho_{\varphi A,t}\rho_{\varphi A,t}^{\intercal}\right)\Sigma_{\varphi,t}^{\intercal}\,\mathrm{d}t}_{\text{remainder}}$$

and the quadratic variance of the $Z_{A,t}$ projection can only equal $\mathrm{d}[\varphi_t, \varphi_t]$ when the remainder quadratic variation is zero. The same arguments also prove the "only if" direction for the second statement when considering the projection to $\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\,\mathrm{d}Z_{A,t}$ instead of $\mathrm{d}Z_{A,t}$.

A.3 Proof of price for terminal payoff, Lemma 3

The price $\Omega(t, X_t, Q_t; T)$ of the claim to a payoff Q_T at terminal date T given the pricing kernel K_t with market price of risk Λ_t corresponds to

$$\Omega(t, X_t, Q_t; T) = \mathbb{E}_t\left[\frac{K_T}{K_t}Q_T\right]$$

and by the law of iterated expectations we have the following recurrence relation for $s \in [t, T]$

$$\Omega(t, X_t, Q_t; T) = \mathbb{E}_t\left[\frac{K_s}{K_t}\mathbb{E}_s\left[\frac{K_T}{K_s}Q_T\right]\right] = \mathbb{E}_t\left[\frac{K_s}{K_t}\Omega(s, X_s, Q_s; T)\right].$$

For a small Δt, we have that

$$\Omega(t, X_t, Q_t; T) = \mathbb{E}_t\left[\frac{K_{t+\Delta t}}{K_t}\Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T)\right].$$

Subtracting $\Omega(t, X_t, Q_t; T)$ from both sides, dividing by Δt and taking $\lim_{\Delta t\downarrow 0}$ we obtain

$$\begin{aligned}0 = &\mathbb{E}_t\left[\lim_{\Delta t\downarrow 0}\frac{e^{-\int_0^{\Delta t} r_{t+s} + \frac{\Lambda_{t+s}^{\intercal}\Lambda_{t+s}}{2}\mathrm{d}s - \int_0^{\Delta t}\Lambda_{t+s}^{\intercal}\mathrm{d}Z_{A,t+s}} - 1}{\Delta t}\Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T)\right]\\ &+ \mathbb{E}_t\left[\lim_{\Delta t\downarrow 0}\frac{\Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) - \Omega(t, X_t, Q_t; T)}{\Delta t}\right]\end{aligned}$$

where

$$
\mathbb{E}_t\left[\lim_{\Delta t\downarrow 0}\frac{e^{-\int_0^{\Delta t} r_{t+s}+\frac{\Lambda_{t+s}^\intercal\Lambda_{t+s}}{2}\mathrm{d}s-\int_0^{\Delta t}\Lambda_{t+s}^\intercal \mathrm{d}Z_{A,t+s}}-1}{\Delta t}\Omega(t+\Delta t,X_{t+\Delta t},Q_{t+\Delta t};T)\right]
$$
$$
=-r_t\Omega(t,X_t,Q_t;T)-(\partial_X\Omega_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\Lambda_t-\partial_Q\Omega_t Q_t\Sigma_{Q,t}\rho_{QA,t}\Lambda_t
$$

and

$$
\mathbb{E}_t\left[\lim_{\Delta t\downarrow 0}\frac{\Omega(t+\Delta t,X_{t+\Delta t},Q_{t+\Delta t};T)-\Omega(t,X_t,Q_t;T)}{\Delta t}\right]
$$
$$
=\partial_t\Omega_t+\partial_Q\Omega_t Q\mu_{Q,t}+(\partial_X\Omega_t)^\intercal\mu_{X,t}+\frac{1}{2}\partial_{QQ}\Omega_t Q_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t+\frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Omega_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
$$
$$
+(\partial_{XQ}\Omega_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal Q_t.
$$

Substituting the expectations and rearranging terms gives the following PDE

$$
\begin{aligned}
0=&-\Omega(t,X_t,Q_t;T)r_t+(\partial_X\Omega_t)^\intercal\left(\mu_{X,t}-\Sigma_{X,t}\rho_{XA,t}\Lambda_t\right)+\left(\mu_{Q,t}-\Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right)Q_t\partial_Q\Omega_t\\
&+\partial_t\Omega_t+\frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Omega_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)+\frac{1}{2}\partial_{QQ}\Omega_t Q_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t+(\partial_{XQ}\Omega_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal Q_t
\end{aligned}
\tag{A.1}
$$

with boundary condition $\Omega(T,X_t,Q_t;T)=Q_t$.

Finally, it's straightforward to reformulate this PDE in terms of $\tilde{\Omega}(t,X_t;T)=\Omega(t,X_t,Q_t;T)Q_t^{-1}$ using homogeneity when market parameters do not depend on Q_t

$$
\begin{aligned}
0=&\partial_t\tilde{\Omega}_t+\left(\mu_{Q,t}-r_t-\Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right)\tilde{\Omega}(t,X_t;T)\\
&+(\partial_X\tilde{\Omega}_t)^\intercal\left(\mu_{X,t}+\Sigma_{X,t}\left(\rho_{XQ,t}\Sigma_{Q,t}^\intercal-\rho_{XA,t}\Lambda_t\right)\right)+\frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\tilde{\Omega}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

with boundary condition $\tilde{\Omega}(T,X_t;T)=1$.

The linear PDE for $\tilde{\Omega}(t,X_t;T)$ is a particular case of the semi-linear PDE from Definition 3 parametrized as

$$
\begin{aligned}
g\Big(&t,X_t;\\
&R_t=\mu_{Q,t}-r_t-\Sigma_{Q,t}\rho_{QA,t}\Lambda_t,\\
&B_t=\mu_{X,t}+\Sigma_{X,t}\left(\rho_{XQ,t}\Sigma_{Q,t}^\intercal-\rho_{XA,t}\Lambda_t\right),\\
&C_t=\Sigma_{X,t}\Sigma_{X,t}^\intercal,\\
&D_t=\Sigma_{X,t}\Sigma_{X,t}^\intercal\Big)
\end{aligned}
\tag{A.2}
$$

with boundary condition $g(T,X_t)=1$.

Section A.21 explains how to reduce the semi-linear PDE to a system of Riccati ODEs when parameters satisfy a quadratic structure in state. In that case, R_t,B_t,C_t,D_t can be constructed from the following building blocks

$$
\begin{aligned}
(\mu_{Q,t}) =& \quad {}_{Q}\alpha + \quad {}_{Q}\beta_p \, X^p + \quad X_p \, \eta_h{}^p \, {}_{Q}\omega^h{}_m \, \eta^m{}_q \, X^q \\
(r_t) =& \quad {}_{r}\alpha + \quad {}_{r}\beta_p \, X^p + \quad X_p \, \eta_h{}^p \, {}_{r}\omega^h{}_m \, \eta^m{}_q \, X^q \\
(\Sigma_{Q,t}\rho_{QA,t}\Lambda_t) =& \quad {}_{\Sigma_Q A}\alpha + \quad {}_{\Sigma_Q A}\beta_p \, X^p + \quad X_p \, \eta_h{}^p \, {}_{\Sigma_Q A}\omega^h{}_m \, \eta^m{}_q \, X^q \\
(\mu_{X,t})^k =& \quad {}_{X}\alpha^k - \quad {}_{X}\beta^k{}_p \, X^p + \quad X_p \, \eta_h{}^p \, {}_{X}\omega^{kh}{}_m \, \eta^m{}_q \, X^q \\
\left(\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\right)^k =& \quad {}_{\Sigma_X\Sigma_Q}\alpha^k + {}_{\Sigma_X\Sigma_Q}\beta^k{}_p \, X^p + X_p \, \eta_h{}^p \, {}_{\Sigma_X\Sigma_Q}\omega^{kh}{}_m \, \eta^m{}_q \, X^q \\
(\Sigma_{X,t}\rho_{XA,t}\Lambda_t)^k =& \quad {}_{\Sigma_X A}\alpha^k + \quad {}_{\Sigma_X A}\beta^k{}_p \, X^p + \quad X_p \, \eta_h{}^p \, {}_{\Sigma_X A}\omega^{kh}{}_m \, \eta^m{}_q \, X^q \\
\left(\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)^k{}_l =& \quad {}_{\Sigma_X}\alpha^k{}_l + \quad {}_{\Sigma_X}\beta^k{}_{lp} \, X^p + \quad X_p \, \eta_h{}^p \, {}_{\Sigma_X}\omega^k{}_l{}^h{}_m \, \eta^m{}_q \, X^q
\end{aligned} \tag{A.3}
$$

Section A.22 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases.

A.4 Proof of replicating strategy for terminal payoff, Lemma 3

Consider the following portfolio value W_t dynamics controlled by investment strategy π_t

$$
\mathrm{d}W_t = W_t \left(r_t \,\mathrm{d}t + \pi_t^\intercal \left((\mu_{A,t} - r_t \mathbf{1}) \,\mathrm{d}t + \Sigma_{A,t} \,\mathrm{d}Z_{A,t} \right) \right) \tag{A.4}
$$

The projection of price Ω_t diffusion terms into tradeable assets is given by

$$
\pi_t = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t} \frac{\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Omega_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Omega_t}{W_t}
$$

resulting in the following dynamics

$$
\begin{aligned}
\mathrm{d}W_t =& W_t r_t \,\mathrm{d}t + \left(\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Omega_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Omega_t\right)^\intercal \Big(\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} (\mu_{A,t} - r_t \mathbf{1}) \,\mathrm{d}t \\
& + \mathcal{P}_{\Sigma_{A,t}^\intercal} \,\mathrm{d}Z_{A,t} \Big)
\end{aligned}
$$

Assuming that this price is replicable (Definition 1), we obtain from (5) that

$$
\begin{aligned}
& \left((\partial_X \Omega_t)^\intercal \Sigma_{X,t} \rho_{XA,t} + \partial_Q \Omega_t Q_t \Sigma_{Q,t} \rho_{QA,t} \right) \mathcal{P}_{\Sigma_{A,t}^\intercal} \,\mathrm{d}Z_{A,t} \\
& = (\partial_X \Omega_t)^\intercal \Sigma_{X,t} \,\mathrm{d}Z_{X,t} + \partial_Q \Omega_t Q_t \Sigma_{Q,t} \,\mathrm{d}Z_{Q,t} \,.
\end{aligned}
$$

Substituting that equivalence and the no arbitrage constraint (3) into portfolio dynamics yields

$$
\begin{aligned}
\mathrm{d}W_t =& W_t r_t \,\mathrm{d}t + \left(\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Omega_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Omega_t\right)^\intercal \mathcal{P}_{\Sigma_{A,t}^\intercal} \Lambda_t \,\mathrm{d}t \\
& + (\partial_X \Omega_t)^\intercal \Sigma_{X,t} \,\mathrm{d}Z_{X,t} + \partial_Q \Omega_t Q_t \Sigma_{Q,t} \,\mathrm{d}Z_{Q,t}
\end{aligned}
$$

Applying Itô's lemma to $\Omega(t, X_t, Q_t; T)$ shows that

$$
\begin{aligned}
\mathrm{d}\Omega_t =& \partial_t \Omega_t \,\mathrm{d}t + (\partial_X \Omega_t)^\intercal \left(\mu_{X,t} \,\mathrm{d}t + \Sigma_{X,t} \,\mathrm{d}Z_{X,t}\right) + \partial_Q \Omega_t Q_t \left(\mu_{Q,t} \,\mathrm{d}t + \Sigma_{Q,t} \,\mathrm{d}Z_{Q,t}\right) \\
& + \frac{1}{2} \operatorname{tr}\left(\partial_{XX^\intercal} \Omega_t \Sigma_{X,t} \Sigma_{X,t}^\intercal\right) \mathrm{d}t + \frac{1}{2} Q_t^2 \partial_{QQ} \Omega_t \Sigma_{Q,t} \Sigma_{Q,t}^\intercal \,\mathrm{d}t + Q_t (\partial_{XQ} \Omega_t)^\intercal \Sigma_{X,t} \rho_{XQ,t} \Sigma_{Q,t}^\intercal \,\mathrm{d}t
\end{aligned} \tag{A.5}
$$

and using the diffusion terms in (A.5) to replace the diffusion terms of the portfolio dynamics yields

$$\begin{aligned}
\mathrm{d}W_t =& \mathrm{d}\Omega_t + W_t r_t \,\mathrm{d}t \\
&- \partial_t \Omega_t \,\mathrm{d}t - (\partial_X \Omega_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t\right)\mathrm{d}t - \partial_Q \Omega_t Q_t \left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t\right)\mathrm{d}t \\
&- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Omega_t \Sigma_{X,t}\Sigma^\intercal_{X,t}\right)\mathrm{d}t - \frac{1}{2}Q_t^2 \partial_{QQ}\Omega_t \Sigma_{Q,t}\Sigma^\intercal_{Q,t}\,\mathrm{d}t - Q_t(\partial_{XQ}\Omega_t)^\intercal \Sigma_{X,t}\rho_{XQ,t}\Sigma^\intercal_{Q,t}\,\mathrm{d}t\,.
\end{aligned}$$

Replacing $\partial_t \Omega_t$ from PDE (A.1) into portfolio dynamics yields

$$\mathrm{d}W_t = \mathrm{d}\Omega_t + W_t r_t \,\mathrm{d}t - \Omega_t r_t \,\mathrm{d}t + \left((\partial_X \Omega_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Omega_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\left(\mathcal{P}_{\Sigma^\intercal_{A,t}} - I\right)\Lambda_t \,\mathrm{d}t\,.$$

Price Ω_t is assumed to be both replicable and spanned by asset risk factors (Lemma 1), so combining (4) and (5) implies that

$$\left((\partial_X \Omega_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Omega_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\left(\mathcal{P}_{\Sigma^\intercal_{A,t}} - I\right) = 0$$

reducing portfolio dynamics to

$$\mathrm{d}W_t = \mathrm{d}\Omega_t + W_t r_t \,\mathrm{d}t - \Omega_t r_t \,\mathrm{d}t\,.$$

For a starting wealth of $W_t = \Omega_t$ we have that $\mathrm{d}W_t = \mathrm{d}\Omega_t$. By induction in following periods $s \in (t, T]$ we have $W_s = \Omega_s$ and $\mathrm{d}W_s = \mathrm{d}\Omega_s$, arriving at $W_T = \Omega_T = Q_T$.

A.5 Proof of price for payoff stream, Lemma 4

The price $\Upsilon(t, X_t, Q_t; T)$ of the claim to a payoff stream Q_t from present time t up to date T given the pricing kernel K_t with market price of risk Λ_t corresponds to the price of a portfolio composed by terminal payoff claims that provides an equivalent payoff stream. By Lemma 3 we have that

$$\Upsilon(t, X_t, Q_t; T) = \mathbb{E}_t\left[\int_t^T \frac{K_u}{K_t} Q_u \,\mathrm{d}u\right] = \int_t^T \Omega(t, X_t, Q_t; u)\,\mathrm{d}u = Q_t \int_t^T \tilde{\Omega}(t, X_t; u)\,\mathrm{d}u\,. \tag{A.6}$$

By the law of iterated expectations we have the following recurrence relation for $s \in [t, T]$

$$\Upsilon(t, X_t, Q_t; T) = \mathbb{E}_t\left[\int_t^s \frac{K_u}{K_t} Q_u \,\mathrm{d}u + \frac{K_s}{K_t}\Upsilon(s, X_s, Q_s; T)\right].$$

For a small Δt, we have that

$$\Upsilon(t, X_t, Q_t; T) = \mathbb{E}_t\left[\int_0^{\Delta t} \frac{K_{t+u}}{K_t} Q_{t+u} \,\mathrm{d}u + \frac{K_{t+\Delta t}}{K_t}\Upsilon(t + \Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T)\right].$$

Subtracting $\Upsilon(t, X_t, Q_t; T)$ from both sides, dividing by Δt and taking $\lim_{\Delta t \downarrow 0}$ we obtain

$$\begin{aligned}
0 = \lim_{\Delta t \downarrow 0} \mathbb{E}_t &\left[\frac{\int_0^{\Delta t} \frac{K_{t+u}}{K_t} Q_{t+u}\,\mathrm{d}u}{\Delta t}\right] + \mathbb{E}_t\left[\frac{\left(\frac{K_{t+\Delta t}}{K_t} - 1\right)}{\Delta t}\Upsilon(t + \Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T)\right] \\
&+ \mathbb{E}_t\left[\frac{\Upsilon(t + \Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) - \Upsilon(t, X_t, Q_t; T)}{\Delta t}\right]
\end{aligned}$$

yielding this PDE

$$\begin{aligned}0 =&\partial_t \Upsilon_t + Q_t - \Upsilon_t r_t + (\partial_X \Upsilon_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\Lambda_t\right) + \partial_Q \Upsilon_t Q_t \left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right)\\ &+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right) + \frac{1}{2}Q_t^2 \partial_{QQ}\Upsilon_t \Sigma_{Q,t}\Sigma_{Q,t}^\intercal + Q_t(\partial_{XQ}\Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\end{aligned} \tag{A.7}$$

with boundary condition $\Upsilon(T, X_t, Q_t; T) = 0$.

Using the Ansatz

$$\Upsilon(t, X_t, Q_t; T) = \int_t^T \Omega(t, X_t, Q_t; s)\,\mathrm{d}s$$

one can see that after substituting it into the PDE

$$\begin{aligned}0 =&Q_t - \overbrace{\Omega(t, X_t, Q_t; t)}^{Q_t}\\ &+ \underbrace{\begin{aligned}\int_t^T \Big(&\partial_t \Omega_{t;s} - \Omega(t, X_t, Q_t; s) r\\ &+ (\partial_X \Omega_{t;s})^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\Lambda_t\right) + Q_t \partial_Q \Omega_{t;s}\left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right)\\ &+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Omega_{t;s}\Sigma_{X,t}\Sigma_{X,t}^\intercal\right) + \frac{1}{2}\partial_{QQ}\Omega_{t;s} Q_t \Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t\\ &\qquad + (\partial_{XQ}\Omega_{t;s})^\intercal \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal Q_t\Big)\,\mathrm{d}s\end{aligned}}_{0}\end{aligned}$$

and notice that the integrand corresponds to the zero condition of the PDE in (A.1) for Ω.

A.6 Proof of replicating strategy for payoff stream, Lemma 4

Consider the following portfolio value W_t dynamics distributing payoff Q_t and controlled by investment strategy π_t

$$\mathrm{d}W_t = - Q_t\,\mathrm{d}t + W_t\left(r_t\,\mathrm{d}t + \pi_t^\intercal\left((\mu_{A,t} - r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\right) \tag{A.8}$$

The projection of price Υ_t diffusion terms into tradeable assets is given by

$$\pi_t = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\frac{\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_X \Upsilon_t + \rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Upsilon_t}{W_t}.$$

resulting in the following portfolio dynamics

$$\begin{aligned}\mathrm{d}W_t =& - Q_t\,\mathrm{d}t + W_t r_t\,\mathrm{d}t\\ &+ \left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\Big(\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\,\mathrm{d}t + \mathcal{P}_{\Sigma_{A,t}^\intercal}\,\mathrm{d}Z_{A,t}\Big)\end{aligned}$$

Assuming that this price is replicable (Definition 1), we obtain from (5) that

$$(\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\,\mathrm{d}Z_{X,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t} = \left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\mathcal{P}_{\Sigma_{A,t}^\intercal}\,\mathrm{d}Z_{A,t}\,.$$

Substituting that equivalence and the no arbitrage constraint (3) into portfolio dynamics yields

$$\begin{aligned}\mathrm{d}W_t =& - Q_t\,\mathrm{d}t + W_t r_t\,\mathrm{d}t + \left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t\,\mathrm{d}t\\ &+ (\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\,\mathrm{d}Z_{X,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t}\end{aligned}$$

Applying Itô's lemma to $\Upsilon(t, X_t, Q_t)$ shows that

$$\begin{aligned}\mathrm{d}\Upsilon_t =&\partial_t \Upsilon_t \,\mathrm{d}t + (\partial_X \Upsilon_t)^\intercal \left(\mu_{X,t}\,\mathrm{d}t + \Sigma_{X,t}\,\mathrm{d}Z_{X,t}\right) + \partial_Q \Upsilon_t Q_t \left(\mu_{Q,t}\,\mathrm{d}t + \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t}\right)\\ &+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\mathrm{d}t + \frac{1}{2}Q_t^2 \partial_{QQ}\Upsilon_t \Sigma_{Q,t}\Sigma_{Q,t}^\intercal\,\mathrm{d}t + Q_t(\partial_{XQ}\Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\,\mathrm{d}t\end{aligned} \tag{A.9}$$

and using the diffusion terms in (A.9) to replace the diffusion terms of the portfolio dynamics yields

$$\begin{aligned}\mathrm{d}W_t =& \mathrm{d}\Upsilon_t - Q_t\,\mathrm{d}t + W_t r_t\,\mathrm{d}t\\ &- \partial_t\Upsilon_t\,\mathrm{d}t - (\partial_X\Upsilon_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t\right)\mathrm{d}t - \partial_Q\Upsilon_t Q_t\left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t\right)\mathrm{d}t\\ &- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\mathrm{d}t - \frac{1}{2}Q_t^2\partial_{QQ}\Upsilon_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal\,\mathrm{d}t - Q_t(\partial_{XQ}\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\,\mathrm{d}t\end{aligned}$$

Replacing $\partial_t\Upsilon_t$ from PDE (A.7) into portfolio dynamics yields

$$\mathrm{d}W_t = \mathrm{d}\Upsilon_t + W_t r_t\,\mathrm{d}t - \Upsilon_t r_t\,\mathrm{d}t + \left((\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XA,t} + \partial_Q\Upsilon_t Q_t\Sigma_{Q,t}\rho_{QA,t}\right)\left(\mathcal{P}_{\Sigma_{A,t}^\intercal} - I\right)\Lambda_t\,\mathrm{d}t$$

Price Υ_t is assumed to be both replicable and spanned by asset risk factors (Lemma 1), so combining (4) and (5) implies that

$$\left((\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XA,t} + \partial_Q\Upsilon_t Q_t\Sigma_{Q,t}\rho_{QA,t}\right)\left(\mathcal{P}_{\Sigma_{A,t}^\intercal} - I\right) = 0$$

reducing portfolio dynamics to

$$\mathrm{d}W_t = \mathrm{d}\Upsilon_t + W_t r_t\,\mathrm{d}t - \Upsilon_t r_t\,\mathrm{d}t\,.$$

For a starting wealth of $W_t = \Upsilon_t$ we have that $\mathrm{d}W_t = \mathrm{d}\Upsilon_t$. By induction in following periods $s \in (t, T]$ we have $W_s = \Upsilon_s$ and $\mathrm{d}W_s = \mathrm{d}\Upsilon_s$, arriving at $W_T = \Upsilon_T = 0$ while distributing Q_t at every instant.

A.7 Proof of returns of payoff claims, Lemma 6

The dynamics of the terminal payoff claim investment $A_{\Omega,t}$ can be derived from (11) by expanding definitions and applying Itô's lemma

$$\begin{aligned}\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} =& \frac{\mathrm{d}\Omega(t, X_t, Q_t; T)}{\Omega(t, X_t, Q_t; T)}\\ =& \frac{\mathrm{d}Q_t}{Q_t} + \frac{\mathrm{d}\tilde{\Omega}(t, X_t; T)}{\tilde{\Omega}(t, X_t; T)} + \frac{(\partial_X\tilde{\Omega}_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal}{\tilde{\Omega}(t, X_t; T)}\,\mathrm{d}t\\ =& \left(\mu_{Q,t} + \frac{\partial_t\tilde{\Omega}_t + (\partial_X\tilde{\Omega}_t)^\intercal\left(\mu_{X,t} + \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\tilde{\Omega}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)}{\tilde{\Omega}(t, X_t; T)}\right)\mathrm{d}t\\ &+ \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t} + \frac{(\partial_X\tilde{\Omega}_t)^\intercal\Sigma_{X,t}}{\tilde{\Omega}(t, X_t; T)}\,\mathrm{d}Z_{X,t}\end{aligned}$$

At this point I replace $\partial_t\tilde{\Omega}_t$ with the value implicitly defined by the PDE (8) and after some simplifications we arrive at

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r_t\,\mathrm{d}t + \Sigma_{Q,t}\left(\rho_{QA,t}\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{Q,t}\right) + \frac{(\partial_X\tilde{\Omega}_t)^\intercal}{\tilde{\Omega}(t, X_t; T)}\Sigma_{X,t}\left(\rho_{XA,t}\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{X,t}\right).$$

where Λ_t is the market price of risk from (3). Further substituting $\mathrm{d}Z_{Q,t}$ by $\rho_{QA,t}\,\mathrm{d}Z_{A,t}$ and $\mathrm{d}Z_{X,t}$ by $\rho_{XA,t}\,\mathrm{d}Z_{A,t}$ yields

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r_t\,\mathrm{d}t + \left(\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal} + \rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\frac{\partial_X\tilde{\Omega}_t}{\tilde{\Omega}(t,X_t;T)}\right)^{\intercal}(\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{A,t})\,.$$

This substitution follows from the assumption that those risk factors are spanned by the market in Lemma 3.

The dynamics of the payoff stream claim investment $A_{\Upsilon,t}$ can be derived from (12). Isolating $A_{\Upsilon,t}$

$$A_{\Upsilon,t} = \frac{\Upsilon(t,X_t,Q_t;T)}{1-\int_{t_0}^{t}\frac{Q_s}{A_{\Upsilon,s}}\,\mathrm{d}s} \tag{A.10}$$

using Itô's lemma

$$\mathrm{d}A_{\Upsilon,t} = -\Upsilon(t,X_t,Q_t;T)\frac{-\frac{Q_t}{A_{\Upsilon,t}}\,\mathrm{d}s}{\left(1-\int_{t_0}^{t}\frac{Q_s}{A_{\Upsilon,s}}\,\mathrm{d}s\right)^2}\,\mathrm{d}t + \frac{1}{1-\int_{t_0}^{t}\frac{Q_s}{A_{\Upsilon,s}}\,\mathrm{d}s}\,\mathrm{d}\Upsilon(t,X_t,Q_t;T)$$

and, at a time point $t\in[t_0,T)$ to avoid division by zero, we arrive at (13) after dividing by (A.10)

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = \frac{Q_t}{\Upsilon(t,X_t,Q_t;T)}\,\mathrm{d}t + \frac{\mathrm{d}\Upsilon(t,X_t,Q_t;T)}{\Upsilon(t,X_t,Q_t;T)}.$$

Expand $\mathrm{d}\Upsilon(t,X_t,Q_t;T)$ using Itô's lemma to obtain

$$\begin{aligned}
\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} =& \frac{Q_t}{\Upsilon(t,X_t,Q_t;T)}\,\mathrm{d}t + \frac{\mathrm{d}Q_t}{Q_t} + \frac{\mathrm{d}\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s} + \frac{\left(\int_t^T\partial_X\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\right)^{\intercal}\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^{\intercal}}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\,\mathrm{d}t\\
=& \frac{\int_t^T\partial_t\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\,\mathrm{d}t + \mu_{Q,t}\,\mathrm{d}t\\
&+ \frac{\left(\int_t^T\partial_X\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\right)^{\intercal}\left(\mu_{X,t}+\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^{\intercal}\right)}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\,\mathrm{d}t\\
&+ \frac{\frac{1}{2}\operatorname{tr}\left(\int_t^T\partial_{XX^{\intercal}}\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\,\Sigma_{X,t}\Sigma_{X,t}^{\intercal}\right)}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\,\mathrm{d}t\\
&+ \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t} + \frac{\left(\int_t^T\partial_X\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\right)^{\intercal}\Sigma_{X,t}}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\,\mathrm{d}Z_{X,t}
\end{aligned}$$

At this point I replace $\partial_t\tilde{\Omega}(t,X_t;s)$ with the value implicitly defined by the PDE (8) and after some simplifications we arrive at

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r_t\,\mathrm{d}t + \Sigma_{Q,t}\left(\rho_{QA,t}\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{Q,t}\right) + \frac{\left(\int_t^T\partial_X\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\right)^{\intercal}\Sigma_{X,t}}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\left(\rho_{XA,t}\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{X,t}\right).$$

Further substituting $\mathrm{d}Z_{Q,t}$ by $\rho_{QA,t}\,\mathrm{d}Z_{A,t}$ and $\mathrm{d}Z_{X,t}$ by $\rho_{XA,t}\,\mathrm{d}Z_{A,t}$ yields

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r_t\,\mathrm{d}t + \left(\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal} + \rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\frac{\int_t^T \partial_X\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}{\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s}\right)^{\intercal}(\Lambda_t\,\mathrm{d}t + \mathrm{d}Z_{A,t})\,.$$

This substitution follows from the assumption that those risk factors are spanned by the market in Lemma 3.

The explicit expression (16) for the cumulative total return of payoff stream claim $A_{\Upsilon,t}$ can be derived from (13). Applying Itô's lemma and for a time point $t \in [t_0, T)$ to avoid division by zero, we arrive at

$$\mathrm{d}\log(A_{\Upsilon,t}) = \frac{Q_t}{\Upsilon(t,X_t,Q_t;T)}\,\mathrm{d}t + \mathrm{d}\log(\Upsilon(t,X_t,Q_t;T))$$

Integrating the equation above with respect to time yields

$$\frac{A_{\Upsilon,t}}{A_{\Upsilon,t_0}} = \frac{\Upsilon(t,X_t,Q_t;T)}{\Upsilon(t_0,X_{t_0},Q_{t_0};T)}e^{\int_{t_0}^t \frac{Q_s}{\Upsilon(s,X_s,Q_s;T)}\mathrm{d}s}. \tag{A.11}$$

A.8 Proof of optimal consumption bundle, Lemma 7

The degenerate case in which the consumption budget is zero $c_t = 0$ admits only one solution allocating zero to every product $c_t\varpi_{i,t} = 0$ and making the maximum amount of consumption bundles consumed zero. In this case the relative allocation decision ϖ_t does not play any role and can be chosen arbitrarily. The remainder of this proof considers the case of a positive consumption budget $c > 0$.

Another degenerate case is when $\theta_i = 0\,\forall i$ or $\exists\theta_i = 1$ make the problem linear in one product or cash-indexed consumption. All other alternatives drop out and the optimal policy allocates the entire budget to the remaining product $\varpi_{i,t} = 1$ or to cash-indexed consumption $1 - \varpi_t^{\intercal}\mathbf{1} = 1$ respectively. The remainder of this proof considers the case of strictly concave problems $\theta \in (0,1)^{n_P}$.

The objective function becomes zero whenever a product with non-zero elasticity is allocated zero consumption, and this applies also to the cash indexed product. This is clearly suboptimal since we can redistribute a fraction of the budget allocated to other products towards products with zero allocations but non-zero elasticities and make the objective function strictly positive. Consequently, we can restrict our attention to non-zero allocations for products that have non-zero elasticities.

These observations make it possible to reformulate the consumption allocation problem (20) in exponential-logarithm form

$$\max_{\varpi_t\in\mathbb{R}_+^{n_P}} \exp\left((1-\theta^{\intercal}\mathbf{1})\log(c_t - c_t\varpi_t^{\intercal}\mathbf{1}) + \sum_{i=1}^{n_P}\theta_i\log\left(\frac{c_t\varpi_{i,t}}{P_{i,t}}\right)\right).$$

Second order conditions are satisfied since the objective function is a monotonic transformation over a strictly concave function. The Hessian matrix of the log-objective is negative definite since it is symmetric and second order (cross-)derivatives are negative. The first order conditions for consumption allocation problem are

$$\varpi_{i,t} =\theta_i \frac{1-\varpi_t^\intercal \mathbf{1}}{1-\theta^\intercal \mathbf{1}}.$$

Aggregating $\varpi_{i,t}$ from the first order conditions we can find this recursive expression for $\varpi_t^\intercal \mathbf{1}$

$$\varpi_t^\intercal \mathbf{1} =\theta^\intercal \mathbf{1} \frac{1-\varpi_t^\intercal \mathbf{1}}{1-\theta^\intercal \mathbf{1}}$$

and rearrange terms to arrive at this other explicit expression

$$\varpi_t^\intercal \mathbf{1} =\theta^\intercal \mathbf{1}.$$

Thus, the optimal allocations to dynamically priced products are given by

$$\varpi_{i,t} = \theta_i,$$

the remainder allocation to cash consumption is

$$1-\varpi_t^\intercal \mathbf{1} = 1-\theta^\intercal \mathbf{1}$$

and the objective function becomes

$$\frac{c_t}{e^{-(1-\theta^\intercal \mathbf{1})\log(1-\theta^\intercal \mathbf{1})-\theta^\intercal \log(\theta)+\theta^\intercal \log(P_t)}}.$$

A.9 Proof of random terminal time, Remark 6

Before addressing the main statement, let me elaborate some auxiliary statements. The probability of survival $S_{m,t|X_{[t,m]}}$ from time t until time $m \in (t,\infty)$ conditional on the hazard rate path $\lambda_{T,[t,m]}$ is

$$S_{m,t|\lambda_{T,[t,m]}} = e^{-\int_t^m \lambda_{T,q}\mathrm{d}q},$$

which may differ from the probability of survival $S_{m,t}$ conditional only on information available at time t, since the termination hazard rate $\lambda_{T,t}$ is adapted to the filtration

$$S_{m,t} = \mathbb{E}_t\left[e^{-\int_t^m \lambda_{T,q}\mathrm{d}q}\right].$$

Conditional on a hazard rate path $\lambda_{T,[t,m]}$, the probability density of terminating at time m from a time t perspective is

$$\partial_m(1-S_{m,t|\lambda_{T,[t,m]}}) = \lambda_{T,m}e^{-\int_t^m \lambda_{T,q}\mathrm{d}q} \tag{A.12}$$

and using (31) the asymptotic probability of termination is

$$1-S_{\infty,t|\lambda_{T,[t,\infty]}} = 1. \tag{A.13}$$

Now consider again the the main statement. Reformulating

$$\mathbb{E}_t\left[\int_t^\tau e^{-\int_t^s \delta_q \mathrm{d}q} u(v(c_s,P_s,\tilde{\theta}))\,\mathrm{d}s\right]$$

in terms of an outer expectation over state paths $\lambda_{T,[t,\infty]}$ and expanding the inner expectation over possible termination points m as an integral using probability density (A.12) produces

$$\mathbb{E}_t\left[\int_t^\infty \lambda_{T,m}e^{-\int_t^m \lambda_{T,q}\mathrm{d}q}\int_t^m \mathbb{E}\left[e^{-\int_t^s \delta_q \mathrm{d}q}u(v(c_s,P_s,\tilde{\theta}))|\mathcal{F}_t,\lambda_{T,[t,\infty]}\right]\mathrm{d}s\,\mathrm{d}m\right]$$

Assuming that (32) converges to a finite value for all possible hazard rate paths $\lambda_{T,[t,\infty]}$ and using integration by parts together with (A.13), the expression becomes

$$\mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s \lambda_{T,q}\mathrm{d}q}\mathbb{E}\left[e^{-\int_t^s \delta_q \mathrm{d}q}u(v(c_s,P_s,\tilde{\theta}))|\mathcal{F}_t,\lambda_{T,[t,\infty]}\right]\mathrm{d}s\right]$$

obtaining

$$\mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s (\lambda_{T,q}+\delta_q)\mathrm{d}q}u(v(c_s,P_s,\tilde{\theta}))\,\mathrm{d}s\right].$$

A.10 Proof of portfolio optimization, Proposition 1

The goal is to maximize

$$\begin{aligned}J(t,W_t,X_t,P_t,Q_t) =& \sup_{\pi,c\in\mathcal{A}}\mathbb{E}_t\left[\varepsilon_1\int_t^T e^{-\int_0^{s-t}\delta_{t+q}\mathrm{d}q}u(v(c_s,P_s,\tilde{\theta}))\,\mathrm{d}s+\varepsilon_2 e^{-\int_0^{T-t}\delta_{t+q}\mathrm{d}q}u(v(W_T,P_T,\theta))\right]\\ &\text{s.t. } \frac{\mathrm{d}W_t}{W_t} = \frac{Q_t\mathbb{1}_{t\le T_R}-c_t}{W_t}\mathrm{d}t+r_t\,\mathrm{d}t+\pi_t^\intercal\left((\mu_{A,t}-r_t\mathbf{1})\,\mathrm{d}t+\Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\end{aligned}$$

for an agent that receives a wage flow in terms of stochastic endowment prices Q_t.

Rewriting the objective function as a recursive function for a small Δt

$$J(t,W_t,X_t,P_t,Q_t) = \sup_{\pi,c\in\mathcal{A}}\mathbb{E}_t\begin{bmatrix}\varepsilon_1 e^{-\int_0^{\Delta t}\delta_{t+q}\mathrm{d}q}u(v(c_t,P_t,\tilde{\theta}))\Delta t\\ +e^{-\int_0^{\Delta t}\delta_{t+q}\mathrm{d}q}J(t+\Delta t,W_{t+\Delta t},X_{t+\Delta t},P_{t+\Delta t},Q_{t+\Delta t})\end{bmatrix}$$

Then subtracting J from both sides, dividing by Δt and taking $\lim_{\Delta t\downarrow 0}$ yields

$$\begin{aligned}0 =& \sup_{\pi,c\in\mathcal{A}}\varepsilon_1\frac{\left(\left(P^\star_{\tilde{\theta},t}\right)^{-1}c_t\right)^{1-\gamma}}{1-\gamma}-\partial_W J_t c_t+\partial_W J_t W_t\pi_t^\intercal(\mu_{A,t}-r_t\mathbf{1})\\ &+\frac{1}{2}\partial_{WW}J_tW_t^2\pi_t^\intercal\Sigma_{A,t}\Sigma_{A,t}^\intercal\pi_t+(\partial_{WX}J_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\Sigma_{A,t}^\intercal\pi_tW_t\\ &+(\partial_{WP}J_t)^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\pi_tW_t\\ &-\delta_tJ(t,W_t,X_t,P_t,Q_t)+\partial_tJ_t+\partial_WJ_t\left(W_tr_t+Q_t\mathbb{1}_{t\le T_R}\right)\\ &+(\partial_XJ_t)^\intercal\mu_{X,t}+\frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}J_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\\ &+(\partial_PJ_t)^\intercal\operatorname{diag}(P_t)\mu_{P,t}+\frac{1}{2}\operatorname{tr}\left(\partial_{PP^\intercal}J_t\operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right)\\ &+\operatorname{tr}\left((\partial_{XP^\intercal}J_t)^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right)\\ &+\partial_QJ_tQ_t\mu_{Q,t}+\frac{1}{2}Q_t^2\partial_{QQ}J_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal+\partial_{WQ}J_tQ_t\Sigma_{Q,t}\rho_{QA,t}\Sigma_{A,t}^\intercal\pi_tW_t\\ &+Q_t(\partial_{XQ}J_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal+(\partial_{QP}J_t)^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PQ,t}\Sigma_{Q,t}^\intercal Q_t\end{aligned}\tag{A.14}$$

with boundary condition $J(T, W_t, X_t, P_t, Q_t) = \varepsilon_2 \left(P^\star_{\theta,t}\right)^{\gamma-1} \frac{W_T^{1-\gamma}}{1-\gamma}$.

Assume that admissibility restrictions other than adaptedness are not locally binding around optimal paths and that indirect utility is increasing and concave in wealth, giving rise to interior solutions. Then the consumption and investment problems are separable. The consumption problem below is concave due to the power utility term

$$\sup_c \varepsilon_1 \frac{\left(\left(P^\star_{\tilde{\theta},t}\right)^{-1} c_t\right)^{1-\gamma}}{1-\gamma} - \partial_W J_t c_t$$

with solution

$$c^\star_t = \varepsilon_1^{\frac{1}{\gamma}} \left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}} (\partial_W J_t)^{-\frac{1}{\gamma}}$$

and using $P^\star_{\tilde{\theta},t}$ from (21) to reduce terms yields (23).

The investment problem is

$$\begin{aligned}\sup_\pi \; & \partial_W J_t W_t \pi_t^\intercal (\mu_{A,t} - r_t \mathbf{1}) + \frac{1}{2} \partial_{WW} J_t W_t^2 \pi_t^\intercal \Sigma_{A,t} \Sigma_{A,t}^\intercal \pi_t + (\partial_{WX} J_t)^\intercal \Sigma_{X,t} \rho_{XA,t} \Sigma_{A,t}^\intercal \pi_t W_t \\ & + (\partial_{WP} J_t)^\intercal \operatorname{diag}(P_t) \Sigma_{P,t} \rho_{PA,t} \Sigma_{A,t}^\intercal \pi_t W_t + \partial_{WQ} J_t Q_t \Sigma_{Q,t} \rho_{QA,t} \Sigma_{A,t}^\intercal \pi_t W_t\end{aligned}$$

with solution

$$\pi^\star_t = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \frac{\begin{aligned}\Big((\mu_{A,t} - r_t\mathbf{1})\partial_W J_t + \Sigma_{A,t}\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_{WX} J_t \\ + \Sigma_{A,t}\rho_{PA,t}^\intercal \Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_{WP} J_t + \Sigma_{A,t}\rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_{WQ} J_t\Big)\end{aligned}}{-\partial_{WW} J_t W_t}.$$

Second order conditions are satisfied given that $\Sigma_{A,t}\Sigma_{A,t}^\intercal$ is positive definite and $\partial_{WW} J_t$ is assumed to be negative, which ultimately follows from diminishing marginal utility of wealth in the utility function. Reformulating partial derivatives produces (24). At $W_t = 0$ the investment problem above is not well defined, instead one should think about it as a prescription for a control $W_t \pi_t$ in terms of nominal wealth exposure, not relative.

$$(W_t\pi_t)^\star = \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \frac{\begin{aligned}\Big((\mu_{A,t} - r_t\mathbf{1})\partial_W J_t + \Sigma_{A,t}\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \partial_{WX} J_t \\ + \Sigma_{A,t}\rho_{PA,t}^\intercal \Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_{WP} J_t + \Sigma_{A,t}\rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_{WQ} J_t\Big)\end{aligned}}{-\partial_{WW} J_t}.$$

Substituting the solutions in the PDE yields

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\gamma}\varepsilon_1^{\frac{1}{\gamma}}\left(P^{\star}_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}}(\partial_W J_t)^{1-\frac{1}{\gamma}} \\
&+\frac{1}{2}\Big((\mu_{A,t}-r_t\mathbf{1})\partial_W J_t \\
&\qquad +\Sigma_{A,t}\rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\partial_{WX}J_t \\
&\qquad +\Sigma_{A,t}\rho_{PA,t}^{\intercal}\Sigma_{P,t}^{\intercal}\operatorname{diag}(P_t)\partial_{WP}J_t \\
&\qquad\qquad +\Sigma_{A,t}\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal}Q_t\partial_{WQ}J_t\Big)^{\intercal}\frac{\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}}{-\partial_{WW}J_t}\Big((\mu_{A,t}-r_t\mathbf{1})\partial_W J_t \\
&\qquad\qquad\qquad +\Sigma_{A,t}\rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\partial_{WX}J_t \\
&\qquad\qquad\qquad +\Sigma_{A,t}\rho_{PA,t}^{\intercal}\Sigma_{P,t}^{\intercal}\operatorname{diag}(P_t)\partial_{WP}J_t \\
&\qquad\qquad\qquad\quad +\Sigma_{A,t}\rho_{QA,t}^{\intercal}\Sigma_{Q,t}^{\intercal}Q_t\partial_{WQ}J_t\Big) \\
&-\delta_t J(t,W_t,X_t,P_t,Q_t)+\partial_t J_t+\partial_W J_t\left(W_t r_t+Q_t\mathbb{1}_{t\le T_R}\right)+(\partial_X J_t)^{\intercal}\mu_{X,t}+\frac{1}{2}\operatorname{tr}\left(\partial_{XX^{\intercal}}J_t\Sigma_{X,t}\Sigma_{X,t}^{\intercal}\right) \\
&+(\partial_P J_t)^{\intercal}\operatorname{diag}(P_t)\mu_{P,t}+\frac{1}{2}\operatorname{tr}\left(\partial_{PP^{\intercal}}J_t\operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^{\intercal}\operatorname{diag}(P_t)\right) \\
&+\operatorname{tr}\left((\partial_{XP^{\intercal}}J_t)^{\intercal}\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^{\intercal}\operatorname{diag}(P_t)\right) \\
&+\partial_Q J_t Q_t\mu_{Q,t}+\frac{1}{2}\partial_{QQ}J_t Q_t^2\Sigma_{Q,t}\Sigma_{Q,t}^{\intercal}+(\partial_{XQ}J_t)^{\intercal}\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^{\intercal}Q_t \\
&+(\partial_{QP}J_t)^{\intercal}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PQ,t}\Sigma_{Q,t}^{\intercal}Q_t
\end{aligned} \tag{A.15}
$$

with boundary condition $J(T,W_t,X_t,P_t,Q_t)=\varepsilon_2\left(P^{\star}_{\theta,t}\right)^{\gamma-1}\frac{W_T^{1-\gamma}}{1-\gamma}$.

A.11 Proof of solution to portfolio optimization, Proposition 2

Continuing from Section A.10, considering the ansatz

$$
J(t,W_t,X_t,P_t,Q_t)=\frac{\left(W_t+\Upsilon(t,X_t,Q_t)\right)^{1-\gamma}}{1-\gamma}f(t,X_t,P_t)^{\gamma}
$$

then the PDE becomes

$$
\begin{aligned}
0 =& \frac{J_t}{f_t}\gamma\Bigg(\partial_t f_t + \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}} \\
&\quad - f_t\left(\frac{\delta_t}{\gamma} - \frac{1-\gamma}{\gamma}\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)\right) \\
&\quad + \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X f_t \\
&\quad + \left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\right)\operatorname{diag}(P_t)\partial_P f_t \\
&\quad + (1-\gamma)\frac{1}{2}\frac{(\partial_X f_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal\partial_X f_t}{f_t} \\
&\quad + (1-\gamma)\frac{1}{2}\frac{(\partial_P f_t)^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P f_t}{f_t} \\
&\quad + (1-\gamma)\frac{(\partial_X f_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - \rho_{XP,t}\right)\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P f_t}{f_t} \\
&\quad + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal} f_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{PP^\intercal} f_t\operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right) \\
&\qquad\qquad + \operatorname{tr}\left((\partial_{XP^\intercal} f_t)^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right)\Bigg) \\
&+ \frac{J_t(1-\gamma)}{W_t+\Upsilon_t}\Bigg(\partial_t\Upsilon_t - \Upsilon_t r_t + Q_t\mathbb{1}_{t\le T_R} \\
&\quad + \left(\mu_{X,t}^\intercal - (\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X\Upsilon_t \\
&\quad + \gamma f_t^{-1}(\partial_X f_t)^\intercal\Sigma_{X,t}\left(I - \rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\right)\Sigma_{X,t}^\intercal\partial_X\Upsilon_t \\
&\quad + \gamma f_t^{-1}(\partial_P f_t)^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\left(\rho_{XP,t}^\intercal - \rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\right)\Sigma_{X,t}^\intercal\partial_X\Upsilon_t \\
&\quad + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right) \\
&\quad + (\partial_{XQ}\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal Q_t \\
&\quad + \left(\mu_{Q,t} - (\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{QA,t}^\intercal\Sigma_{Q,t}^\intercal\right)Q_t\partial_Q\Upsilon_t \\
&\quad + \gamma f_t^{-1}(\partial_X f_t)^\intercal\Sigma_{X,t}\left(\rho_{XQ,t} - \rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{QA,t}^\intercal\right)\Sigma_{Q,t}^\intercal Q_t\partial_Q\Upsilon_t \\
&\quad + \gamma f_t^{-1}(\partial_P f_t)^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\left(\rho_{PQ,t} - \rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{QA,t}^\intercal\right)\Sigma_{Q,t}^\intercal Q_t\partial_Q\Upsilon_t \\
&\quad + \frac{1}{2}\partial_{QQ}\Upsilon_t Q_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t \\
&\quad + \frac{1}{W_t+\Upsilon_t}\gamma\frac{1}{2}(\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal\partial_X\Upsilon_t \\
&\quad + \frac{1}{W_t+\Upsilon_t}\gamma(\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{QA,t}^\intercal - \rho_{XQ,t}\right)\Sigma_{Q,t}^\intercal Q_t\partial_Q\Upsilon_t \\
&\qquad + \frac{1}{W_t+\Upsilon_t}\gamma\frac{1}{2}\partial_Q\Upsilon_t Q_t\Sigma_{Q,t}\left(\rho_{QA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{QA,t}^\intercal - I\right)\Sigma_{Q,t}^\intercal Q_t\partial_Q\Upsilon_t\Bigg)
\end{aligned}
\tag{A.16}
$$

The terms multiplying $\frac{J_t}{f_t}\gamma$ in the PDE above must add up to zero as well as the terms multiplying $\frac{J_t(1-\gamma)}{W_t+\Upsilon_t}$. These condition gives rise to PDEs for $f(t,X_t,P_t)$ and $\Upsilon(t,X_t,Q_t)$. The associated optimal controls are

$$c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} \left(P_{\tilde{\theta},t}^\star\right)^{1-\frac{1}{\gamma}} f(t,X_t,P_t)^{-1} \left(W_t + \Upsilon(t,X_t,Q_t)\right) \tag{A.17}$$

$$\begin{aligned}
\pi_t^\star &= \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \frac{(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} \\
&\quad + \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t}\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \left(f(t,X_t,P_t)^{-1}\partial_X f_t \frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} - \frac{\partial_X \Upsilon_t}{W_t}\right) \\
&\quad + \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t}\rho_{PA,t}^\intercal \Sigma_{P,t}^\intercal \operatorname{diag}(P_t) f(t,X_t,P_t)^{-1}\partial_P f_t \frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} \\
&\quad - \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t}\rho_{QA,t}^\intercal \Sigma_{Q,t}^\intercal Q_t \partial_Q \Upsilon_t \frac{1}{W_t}
\end{aligned} \tag{A.18}$$

which coincide with (26) and when using $P_{\tilde{\theta},t}^\star$ from (21) to reduce terms and reformulating partial derivatives to obtain (27).

Solution to $f(t,X_t,P_t)$. The terms multiplying $\frac{J_t}{f_t}\gamma$ in (A.16) give rise to a PDE for $f(t,X_t,P_t)$

$$\begin{aligned}
0 =& \varepsilon_1^{\frac{1}{\gamma}} \left(P_{\tilde{\theta},t}^\star\right)^{1-\frac{1}{\gamma}} + \partial_t f_t \\
& - f(t,X_t,P_t) \frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma} \\
& + \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t}\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \right) \partial_X f_t \\
& + \left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t}\rho_{PA,t}^\intercal \Sigma_{P,t}^\intercal \right) \operatorname{diag}(P_t)\partial_P f_t \\
& + \frac{1}{2}(1-\gamma)\frac{(\partial_X f_t)^\intercal \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal \partial_X f_t}{f(t,X_t,P_t)} \\
& + \frac{1}{2}(1-\gamma)\frac{(\partial_P f_t)^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_P f_t}{f(t,X_t,P_t)} \\
& + (1-\gamma)\frac{(\partial_X f_t)^\intercal \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - \rho_{XP,t}\right)\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_P f_t}{f(t,X_t,P_t)} \\
& + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal} f_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right) \\
& + \frac{1}{2}\operatorname{tr}\left(\partial_{PP^\intercal} f_t \operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\right) \\
& + \operatorname{tr}\left((\partial_{XP^\intercal} f_t)^\intercal \Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\right)
\end{aligned} \tag{A.19}$$

with boundary condition $f(T,X_t,P_t) = \varepsilon_2^{\frac{1}{\gamma}}\left(e^{(1-\theta^\intercal\mathbf{1})\log(1-\theta^\intercal\mathbf{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P_T)}\right)^{\frac{1}{\gamma}-1}$.

There are three explicit solutions depending on ε_1 and the degree of market completeness. If there is no intermediate consumption $\varepsilon_1 = 0$, one can use the Ansatz

$$f(t,X_t,P_t) = \varepsilon_2^{\frac{1}{\gamma}}\left(P_{\theta,t}^\star\right)^{1-\frac{1}{\gamma}} h(t,X_t;T,\theta) \tag{A.20}$$

then

$$
\begin{aligned}
\partial_t h_t =h_t\Bigg(\frac{\delta_t}{\gamma} - \left(\frac{1}{\gamma}-1\right) \Bigg(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \\
&- \theta^\intercal \left(\mu_{P,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})\right) \\
&+ \frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_{P,t}\left(\gamma I + (1-\gamma)\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta \\
&+\frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\Bigg)\Bigg) \\
&- (\partial_X h_t)^\intercal\Bigg(\mu_{X,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{X,t}\Bigg(\rho_{XA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1}) \\
&- \left(\gamma\rho_{XP,t} + (1-\gamma)\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta\Bigg)\Bigg) \\
&- \frac{1}{2}(1-\gamma)\frac{(\partial_X h_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal\partial_X h_t}{h_t} \\
&- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}h_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
\tag{A.21}
$$

with boundary condition $h(T, X_t; T, \theta) = 1$.

The PDE for $h(t, X_t; T, \theta)$ is a particular case of the semi-linear PDE from Definition 3 parametrized as

$$
\begin{aligned}
&g\Bigg(t, X_t; \\
&R_t = -\frac{\delta_t}{\gamma} + \frac{1-\gamma}{\gamma}\Bigg(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \\
&\quad - \theta^\intercal\left(\mu_{P,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})\right) \\
&\quad + \frac{1}{2}\frac{1-\gamma}{\gamma}\theta^\intercal\Sigma_{P,t}\left(\gamma I + (1-\gamma)\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta \\
&\quad +\frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\Bigg), \\
&B_t = \mu_{X,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{X,t}\Bigg(\rho_{XA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1}) \\
&\quad - \left(\gamma\rho_{XP,t} + (1-\gamma)\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta\Bigg), \\
&C_t = \Sigma_{X,t}\left(\gamma I + (1-\gamma)\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\right)\Sigma_{X,t}^\intercal, \\
&D_t = \Sigma_{X,t}\Sigma_{X,t}^\intercal\Bigg)
\end{aligned}
\tag{A.22}
$$

with boundary condition $g(T, X_t) = 1$.

The matrices C_t and D_t are symmetric and at least positive semi-definite. In the case of C_t, notice that when formulated in this way

$$C_t = \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t} + \gamma\left(I - \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}\right)\right)\Sigma^\intercal_{X,t}$$

we only need to show that $I - \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}$ is positive semi-definite. Since the each standard Brownian motion vector $Z_{X,t}$ and $Z_{A,t}$ is composed of orthogonal components, auto-covariance and auto-correlation coincide, e.g. $\rho_{XX,t} = I$, and $Z_{X,t}$ can be decomposed using its projection to $Z_{A,t}$ and a complementary orthogonal standard Brownian motion vector $Z_{\bar{A},t}$

$$\rho_{XX,t} = I = \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t} + \rho_{X\bar{A},t}\rho^\intercal_{X\bar{A},t}.$$

Thus

$$I - \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t} = \rho_{X\bar{A},t}\rho^\intercal_{X\bar{A},t}$$

is positive semi-definite.

Section A.21 explains how to reduce the semi-linear PDE to a system of Riccati ODEs when parameters satisfy a quadratic structure in state. In that case, R_t, B_t, C_t, D_t can be constructed from the following building blocks

$$\begin{array}{rcrrr}
(\delta_t) &=& {}_{\delta}\alpha + & {}_{\delta}\beta_p\, X^p + & X_p\, \eta_h{}^p\, {}_{\delta}\omega^h{}_m\, \eta^m{}_q\, X^q \\
(r_t) &=& {}_{r}\alpha + & {}_{r}\beta_p\, X^p + & X_p\, \eta_h{}^p\, {}_{r}\omega^h{}_m\, \eta^m{}_q\, X^q \\
(\Lambda_t^\intercal \mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t) &=& {}_{A}\tilde{\alpha} + & {}_{A}\tilde{\beta}_p\, X^p + & X_p\, \eta_h{}^p\, {}_{A}\tilde{\omega}^h{}_m\, \eta^m{}_q\, X^q \\
(\mu_{P,t})^k &=& {}_{P}\alpha^k + & {}_{P}\beta^k{}_p\, X^p + & X_p\, \eta_h{}^p\, {}_{P}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t\right)^k &=& {}_{\Sigma_P A}\alpha^k + & {}_{\Sigma_P A}\beta^k{}_p\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_P A}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\right)^k{}_l &=& {}_{\Sigma_P}\tilde{\alpha}^k{}_l + & {}_{\Sigma_P}\tilde{\beta}^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_P}\tilde{\omega}^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{P,t}\Sigma^\intercal_{P,t}\right)^k{}_l &=& {}_{\Sigma_P}\alpha^k{}_l + & {}_{\Sigma_P}\beta^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_P}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
(\mu_{X,t})^k &=& {}_{X}\alpha^k - & {}_{X}\beta^k{}_p\, X^p + & X_p\, \eta_h{}^p\, {}_{X}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t\right)^k &=& {}_{\Sigma_X A}\alpha^k + & {}_{\Sigma_X A}\beta^k{}_p\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_X A}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{X,t}\rho_{XP,t}\Sigma^\intercal_{P,t}\right)^k{}_l &=& {}_{\Sigma_X\Sigma_P}\alpha^k{}_l + & {}_{\Sigma_X\Sigma_P}\beta^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_X\Sigma_P}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\right)^k{}_l &=& {}_{\Sigma_X\Sigma_P}\tilde{\alpha}^k{}_l + & {}_{\Sigma_X\Sigma_P}\tilde{\beta}^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_X\Sigma_P}\tilde{\omega}^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\right)^k{}_l &=& {}_{\Sigma_X}\tilde{\alpha}^k{}_l + & {}_{\Sigma_X}\tilde{\beta}^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_X}\tilde{\omega}^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_{X,t}\Sigma^\intercal_{X,t}\right)^k{}_l &=& {}_{\Sigma_X}\alpha^k{}_l + & {}_{\Sigma_X}\beta^k{}_{lp}\, X^p + & X_p\, \eta_h{}^p\, {}_{\Sigma_X}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q
\end{array} \tag{A.23}$$

where

$$\begin{aligned}
\Lambda_t^\intercal\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t &= (\mu_{A,t} - r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}(\mu_{A,t} - r_t\mathbf{1}) \\
\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t &= \Sigma_{P,t}\rho_{PA,t}\Sigma^\intercal_{A,t}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}(\mu_{A,t} - r_t\mathbf{1}) \\
\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\Lambda_t &= \Sigma_{X,t}\rho_{XA,t}\Sigma^\intercal_{A,t}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})
\end{aligned}$$

by the no arbitrage constraint (3).

Section A.22 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases.

An explicit solution with intermediate consumption $\varepsilon_1 \neq 0$ arises when assuming that either $\gamma \to 1$ or that markets are complete enough to satisfy at all times

$$
\begin{aligned}
0 =& \frac{1}{2}(1-\gamma)\frac{(\partial_X f_t)^\intercal \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal \partial_X f_t}{f(t,X_t,P_t)} \\
&+ \frac{1}{2}(1-\gamma)\frac{(\partial_P f_t)^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_P f_t}{f(t,X_t,P_t)} \\
&+ (1-\gamma)\frac{(\partial_X f_t)^\intercal \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - \rho_{XP,t}\right)\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\partial_P f_t}{f(t,X_t,P_t)}
\end{aligned} \tag{A.24}
$$

By Lemma 2, the second clause requires $f(t, X_t, P_t)$ to be replicable when $\gamma \neq 1$. Now for this case consider the following ansatz

$$
f(t,X_t,P_t) = \varepsilon_2^{\frac{1}{\gamma}}\left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}}\hat{h}(t,X_t;T) + \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}}\int_t^T \tilde{h}(t,X_t;s)\,\mathrm{d}s \tag{A.25}
$$

we can replace them in PDE (A.19) to arrive at

$$
\begin{aligned}
0 =& \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}} \\
&\cdot \Bigg(1 - \overbrace{\tilde{h}(t,X_t;t)}^{1} \\
&\quad \underbrace{- \int_t^T \Bigg(- \partial_t \tilde{h}_{t;s}
+ \tilde{h}_{t;s}\frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma} \\
&\qquad + \tilde{h}_{t;s}\left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\right)\tilde{\theta}\left(\frac{1}{\gamma}-1\right) \\
&\qquad - \tilde{h}_{t;s}\frac{1}{2}\left(\frac{1}{\gamma}-1\right)^2\tilde{\theta}^\intercal\Sigma_{P,t}\Sigma_{P,t}^\intercal\tilde{\theta} \\
&\qquad - \tilde{h}_{t;s}\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\operatorname{tr}\left(\operatorname{diag}\left(\tilde{\theta}\right)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right) \\
&\qquad - \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X\tilde{h}_{t;s} \\
&\qquad\quad + \left(\frac{1}{\gamma}-1\right)(\partial_X\tilde{h}_{t;s})^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\tilde{\theta} - \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\tilde{h}_{t;s}\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\Bigg)\mathrm{d}s}_{0}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon_2^{\frac{1}{\gamma}} \left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}} \Bigg(- \partial_t \hat{h}_t \\
&\qquad + \hat{h}_t \frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma} \\
&\qquad + \hat{h}_t \left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\right)\theta\left(\frac{1}{\gamma}-1\right) \\
&\qquad - \hat{h}_t \frac{1}{2}\left(\frac{1}{\gamma}-1\right)^2 \theta^\intercal \Sigma_{P,t}\Sigma_{P,t}^\intercal \theta \\
&\qquad - \hat{h}_t \frac{1}{2}\left(\frac{1}{\gamma}-1\right)\operatorname{tr}\left(\operatorname{diag}(\theta)\,\Sigma_{P,t}\Sigma_{P,t}^\intercal\right) \\
&\qquad - \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X \hat{h}_t \\
&\qquad\qquad + \left(\frac{1}{\gamma}-1\right)(\partial_X\hat{h}_t)^\intercal \Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\theta - \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\hat{h}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\Bigg)
\end{aligned}
$$

We can see that, on the one hand the terms multiplying $\varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}}$ must be zero, and on the other hand the terms multiplying $\varepsilon_2^{\frac{1}{\gamma}}\left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}}$ must be zero, for the PDE to hold for any P_t, $\tilde{\theta}$ and θ. Thus, we can reformulate the former PDE into this one for $\tilde{h}(t, X_t; T)$

$$
\begin{aligned}
\partial_t \tilde{h}_t =& \tilde{h}_t \Bigg(\frac{\delta_t}{\gamma} - \left(\frac{1}{\gamma}-1\right)\Bigg(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \\
&\qquad - \tilde{\theta}^\intercal\left(\mu_{P,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})\right) \\
&\qquad\qquad + \frac{1}{2}\left(\frac{1}{\gamma}-1\right)\tilde{\theta}^\intercal\Sigma_{P,t}\Sigma_{P,t}^\intercal\tilde{\theta} + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}\left(\tilde{\theta}\right)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\Bigg)\Bigg) \\
&- (\partial_X\tilde{h}_t)^\intercal\left(\mu_{X,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{X,t}\left(\rho_{XA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1}) - \rho_{XP,t}\Sigma_{P,t}^\intercal\tilde{\theta}\right)\right) \\
&- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\tilde{h}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

with boundary condition $\tilde{h}(s, X_t; s) = 1$ at any terminal date $s \in [t, T]$, and this other one for $\hat{h}(t, X_t; T)$

$$
\begin{aligned}
\partial_t \hat{h}_t =& \hat{h}_t \Bigg(\frac{\delta_t}{\gamma} - \left(\frac{1}{\gamma}-1\right)\Bigg(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \\
&\qquad - \theta^\intercal\left(\mu_{P,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})\right) \\
&\qquad\qquad + \frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_{P,t}\Sigma_{P,t}^\intercal\theta + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\,\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\Bigg)\Bigg) \\
&- (\partial_X\hat{h}_t)^\intercal\left(\mu_{X,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{X,t}\left(\rho_{XA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1}) - \rho_{XP,t}\Sigma_{P,t}^\intercal\theta\right)\right) \\
&- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\hat{h}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

with boundary condition $\hat{h}(T, X_t; T) = 1$. Both of these PDEs are particular cases of $h(t, X_t; T, \theta)$ from (A.21) inheriting its closed form solutions. They just use different parameter values and are conditioned by the restrictions assumed at (A.24).

Another explicit solution with intermediate consumption $\varepsilon_1 \neq 0$ arises when assuming that $\tilde{\theta} = \theta$ and markets are complete enough to satisfy at all times

$$0 = \frac{1}{2}(1-\gamma)\frac{(\partial_X f_t)^\intercal \Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal - I\right)\Sigma_{X,t}^\intercal \partial_X f_t}{f(t, X_t, P_t)} \tag{A.26}$$

By Lemma 2, the second clause requires $(\partial_X f_t)^\intercal \Sigma_{X,t}\,\mathrm{d}Z_{X,t}$ be replicable when $\gamma \neq 1$. Now for this case consider the following ansatz

$$f(t, X_t, P_t) = \left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}}\left(\varepsilon_2^{\frac{1}{\gamma}}\check{h}(t, X_t; T) + \varepsilon_1^{\frac{1}{\gamma}}\int_t^T \check{h}(t, X_t; s)\,\mathrm{d}s\right), \tag{A.27}$$

we can replace it in PDE (A.19) to arrive at

$$\begin{aligned}
0 =& \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}} \\
&\cdot \Bigg(1 - \overbrace{\check{h}(t, X_t; t)}^{1} \\
&\quad \underbrace{\begin{aligned} - \int_t^T \Bigg(& -\partial_t \check{h}_{t;s} \\
& + \check{h}_{t;s}\frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma} \\
& + \check{h}_{t;s}\left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\right)\theta\left(\frac{1}{\gamma}-1\right) \\
& - \check{h}_{t;s}\frac{1}{2}\left(\frac{1}{\gamma}-1\right)^2\theta^\intercal\Sigma_{P,t}\Sigma_{P,t}^\intercal\theta \\
& - \check{h}_{t;s}\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\operatorname{tr}\left(\operatorname{diag}(\theta)\,\Sigma_{P,t}\Sigma_{P,t}^\intercal\right) \\
& - \check{h}_{t;s}\frac{1}{2}(1-\gamma)\left(\frac{1}{\gamma}-1\right)^2\theta^\intercal\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal\theta \\
& - \check{h}_{t;s}\frac{1}{2}(1-\gamma)\left(\frac{1}{\gamma}-1\right)^2\theta^\intercal\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal\theta \\
& + (1-\gamma)\left(\frac{1}{\gamma}-1\right)(\partial_X\check{h}_{t;s})^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - \rho_{XP,t}\right)\Sigma_{P,t}^\intercal\theta \\
& - \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X\check{h}_{t;s} \\
& \qquad + \left(\frac{1}{\gamma}-1\right)(\partial_X\check{h}_{t;s})^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\theta - \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\check{h}_{t;s}\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\Bigg)\,\mathrm{d}s\Bigg)\end{aligned}}_{0}
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon_2^{\frac{1}{\gamma}} \left(P^\star_{\theta,t}\right)^{1-\frac{1}{\gamma}} \Bigg(- \partial_t \check{h}_t \\
&\qquad + \check{h}_t \frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma} \\
&\qquad + \check{h}_t \left(\mu_{P,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\right)\theta\left(\frac{1}{\gamma}-1\right) \\
&\qquad - \check{h}_t \frac{1}{2}\left(\frac{1}{\gamma}-1\right)^2 \theta^\intercal \Sigma_{P,t}\Sigma_{P,t}^\intercal \theta \\
&\qquad - \check{h}_t \frac{1}{2}\left(\frac{1}{\gamma}-1\right) \operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right) \\
&\qquad - \check{h}_t \frac{1}{2}(1-\gamma)\left(\frac{1}{\gamma}-1\right)^2 \theta^\intercal\Sigma_{P,t}\left(\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - I\right)\Sigma_{P,t}^\intercal\theta \\
&\qquad + (1-\gamma)\left(\frac{1}{\gamma}-1\right)(\partial_X\check{h}_t)^\intercal\Sigma_{X,t}\left(\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal - \rho_{XP,t}\right)\Sigma_{P,t}^\intercal\theta \\
&\qquad - \left(\mu_{X,t}^\intercal + \frac{1-\gamma}{\gamma}(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\right)\partial_X\check{h}_t \\
&\qquad\qquad + \left(\frac{1}{\gamma}-1\right)(\partial_X\check{h}_t)^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\theta - \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\check{h}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\Bigg)
\end{aligned}
$$

We can see that terms multiplying $\varepsilon_1^{\frac{1}{\gamma}}$ and $\varepsilon_2^{\frac{1}{\gamma}}$ must be zero, and both imply identical dynamics for $\check{h}(t, X_t; s)$. Thus, we can reformulate the former PDE into a PDE for $\check{h}(t, X_t; T)$

$$
\begin{aligned}
\partial_t \check{h}_t = &\check{h}_t \Bigg(\frac{\delta_t}{\gamma} - \left(\frac{1}{\gamma}-1\right)\Bigg(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma} \\
&\qquad - \theta^\intercal\left(\mu_{P,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})\right) \\
&\qquad + \frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_{P,t}\left(\gamma I + (1-\gamma)\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta \\
&\qquad\qquad + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\Bigg)\Bigg) \\
&- (\partial_X\check{h}_t)^\intercal\Bigg(\mu_{X,t} + \left(\frac{1}{\gamma}-1\right)\Sigma_{X,t}\Bigg(\rho_{XA,t}\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t}-r_t\mathbf{1}) \\
&\qquad - \left(\gamma\rho_{XP,t} + (1-\gamma)\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\right)\Sigma_{P,t}^\intercal\theta\Bigg)\Bigg) \\
&- \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\check{h}_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

with boundary condition $\check{h}(s, X_t; s) = 1$ at any terminal date $s \in [t, T]$, which is just a particular case of $h(t, X_t; T, \theta)$ from (A.21) inheriting its closed form solutions. The only differences is that it is conditioned by the restrictions assumed at (A.26).

Solution to $\Upsilon(t, X_t, Q_t)$. The terms multiplying $\frac{J_t(1-\gamma)}{W_t+\Upsilon_t}$ in (A.16) have a trivial solution when $Q_t = 0$ at all times, making $\Upsilon(t, X_t, Q_t) = 0$. Otherwise, they can give rise to a PDE for

$\Upsilon(t, X_t, Q_t)$ under appropriate conditions.

To remove dependencies on W_t, markets need to be complete enough to satisfy at all times

$$\begin{aligned}
0 =& \frac{1}{2}(\partial_X \Upsilon_t)^\intercal \Sigma_{X,t} \left(\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t} - I\right) \Sigma^\intercal_{X,t}\partial_X \Upsilon_t \\
&+ (\partial_X \Upsilon_t)^\intercal \Sigma_{X,t} \left(\rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{QA,t} - \rho_{XQ,t}\right) \Sigma^\intercal_{Q,t} Q_t \partial_Q \Upsilon_t \\
&+ \frac{1}{2}\partial_Q \Upsilon_t Q_t \Sigma_{Q,t} \left(\rho_{QA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{QA,t} - I\right) \Sigma^\intercal_{Q,t} Q_t \partial_Q \Upsilon_t.
\end{aligned} \tag{A.28}$$

By Lemma 2, this condition requires Υ_t to be replicable.

To remove dependencies on P_t, markets need to be complete enough to satisfy at all times

$$\begin{aligned}
0 =& \frac{(\partial_X f_t)^\intercal}{f_t} \Sigma_{X,t} \left(I - \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}\right) \Sigma^\intercal_{X,t}\partial_X \Upsilon_t \\
&+ \frac{(\partial_P f_t)^\intercal}{f_t} \operatorname{diag}(P_t)\Sigma_{P,t} \left(\rho^\intercal_{XP,t} - \rho_{PA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}\right) \Sigma^\intercal_{X,t}\partial_X \Upsilon_t \\
&+ \frac{(\partial_X f_t)^\intercal}{f_t} \Sigma_{X,t} \left(\rho_{XQ,t} - \rho_{XA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{QA,t}\right) \Sigma^\intercal_{Q,t} Q_t \partial_Q \Upsilon_t \\
&+ \frac{(\partial_P f_t)^\intercal}{f_t} \operatorname{diag}(P_t)\Sigma_{P,t} \left(\rho_{PQ,t} - \rho_{PA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{QA,t}\right) \Sigma^\intercal_{Q,t} Q_t \partial_Q \Upsilon_t
\end{aligned} \tag{A.29}$$

Since Υ_t is assumed to be replicable, it can be rewritten as the expression below and the multiplication against the asset market residual makes it zero

$$\begin{aligned}
0 = & \underbrace{\left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + (\partial_Q \Upsilon_t) Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\left(I - \mathcal{P}_{\Sigma^\intercal_{A,t}}\right)}_{0} \Bigg(\frac{(\partial_X f_t)^\intercal}{f_t}\Sigma_{X,t}\rho_{XA,t} \\
&\qquad + \frac{(\partial_P f_t)^\intercal}{f_t} \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\Bigg)^\intercal
\end{aligned}$$

Under these conditions the PDE for $\Upsilon(t, X_t, Q_t)$ simplifies to

$$\begin{aligned}
0 =& \partial_t \Upsilon_t - \Upsilon(t, X_t, Q_t) r_t + Q_t \mathbb{1}_{t \le T_R} \\
&+ (\partial_X \Upsilon_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\Sigma^\intercal_{A,t}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}(\mu_{A,t} - r_t \mathbf{1})\right) \\
&+ \partial_Q \Upsilon_t Q_t \left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\Sigma^\intercal_{A,t}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}(\mu_{A,t} - r_t \mathbf{1})\right) \\
&+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t \Sigma_{X,t}\Sigma^\intercal_{X,t}\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{QQ^\intercal}\Upsilon_t Q_t \Sigma_{Q,t}\Sigma^\intercal_{Q,t} Q_t\right) \\
&+ (\partial_{XQ}\Upsilon_t)^\intercal Q_t \Sigma_{X,t}\rho_{XQ,t}\Sigma^\intercal_{Q,t}
\end{aligned} \tag{A.30}$$

with boundary condition $\Upsilon(T, X_t, Q_t) = 0$.

The time derivative implied by this PDE has two stages, one for $t \le T_R$ and another for $t > T_R$. At time T the boundary condition $\Upsilon(T, X_t, Q_t) = 0$ implies that $\partial_t \Upsilon_t|_{t=T} = 0$. By backwards induction we can see that this relation holds steady during the second stage $t \in (T_R, T]$, making Υ zero and in particular $\Upsilon(T_R, X_{T_R}, Q_{T_R}) = 0$. Thus $\Upsilon(t, X_t, Q_t)$ is the solution to the first stage PDE

$$
\begin{aligned}
0 =& \partial_t \Upsilon_t - \Upsilon(t, X_t, Q_t) r_t + Q_t \\
&+ (\partial_X \Upsilon_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\right) \\
&+ \partial_Q \Upsilon_t Q_t \left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\right) \\
&+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{QQ^\intercal}\Upsilon_t Q_t \Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t\right) \\
&+ (\partial_{XQ}\Upsilon_t)^\intercal Q_t \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal
\end{aligned}
$$

with boundary condition $\Upsilon(T_R, X_t, Q_t) = 0$ and it stays at zero thereafter $t \in (T_R, T]$.

By the no arbitrage constraint (3)

$$
\begin{aligned}
&\left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1}) \\
&= \left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t.
\end{aligned}
$$

Price Υ_t is assumed to be both replicable and spanned by asset risk factors (Lemma 1), so combining (4) and (5) implies that

$$
\left((\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\right)\mathcal{P}_{\Sigma_{A,t}^\intercal} = (\partial_X \Upsilon_t)^\intercal \Sigma_{X,t}\rho_{XA,t} + \partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}
$$

making the PDE equal to

$$
\begin{aligned}
0 =& \partial_t \Upsilon_t - \Upsilon_t r_t + Q_t + (\partial_X \Upsilon_t)^\intercal \left(\mu_{X,t} - \Sigma_{X,t}\rho_{XA,t}\Lambda_t\right) + \partial_Q \Upsilon_t Q_t \left(\mu_{Q,t} - \Sigma_{Q,t}\rho_{QA,t}\Lambda_t\right) \\
&+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{QQ^\intercal}\Upsilon_t Q_t \Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t\right) + (\partial_{XQ}\Upsilon_t)^\intercal Q_t \Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal.
\end{aligned}
$$

This solution coincides with the payoff stream $\Upsilon(t, X_t, Q_t; T_R)$ in (A.7) from Lemma 4 and inherits its closed form solutions. Note that the equivalence $Q_t \partial_Q \Upsilon_t = \Upsilon(t, X_t, Q_t; T_R)$ from Lemma 4 can be replaced into (A.18), obtaining (27).

A.12 Proof of future consumption price in complete markets, Remark 3

In Proposition 2 when markets are complete enough to replicate $f(t, X_t, P_t)$ and $h(t, X_t; T, \theta)$, terms (A.24) dissappear

$$
\begin{aligned}
0 =& \frac{1}{2} f_t \left(\frac{(\partial_X f_t)^\intercal}{f_t}\Sigma_{X,t}\rho_{XA,t} + \frac{(\partial_P f_t)^\intercal}{f_t}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\right) \\
&\cdot \left(\mathcal{P}_{\Sigma_{A,t}^\intercal} - I\right)\left(\rho_{XA,t}^\intercal \Sigma_{X,t}^\intercal \frac{\partial_X f_t}{f_t} + \rho_{PA,t}^\intercal \Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\frac{\partial_P f_t}{f_t}\right)
\end{aligned}
$$

and PDE (A.21) for $h(t, X_t; T, \theta)$ becomes

$$
\begin{aligned}
0 =& \partial_t h_t + h_t \left(\mu_{\hat{Q},t} - r_t - \Sigma_{\hat{Q},t}\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\right) \\
&+ (\partial_X h_t)^\intercal \left(\mu_{X,t} + \Sigma_{X,t}\left(\rho_{XA,t}\Sigma_{\hat{Q},t}^\intercal - \rho_{XA,t}\Sigma_{A,t}^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\right)\right) \\
&+ \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal} h_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_{\hat{Q},t} =& \frac{1}{\gamma}(\mu_{A,t} - r_t \mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1} \Sigma_{A,t} - \frac{1-\gamma}{\gamma}\theta\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal} \\
\mu_{\hat{Q},t} =& -\frac{\delta_t}{\gamma} + \frac{1}{\gamma}\left(r_t + \frac{1}{2}(\mu_{A,t} - r_t\mathbf{1})^\intercal \left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}(\mu_{A,t} - r_t\mathbf{1})\right) \\
& - \frac{1-\gamma}{\gamma}\left(\theta^\intercal \mu_{P,t} - \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\,\Sigma_{P,t}\Sigma_{P,t}^\intercal\right)\right) + \frac{1}{2}\Sigma_{\hat{Q},t}\Sigma_{\hat{Q},t}^\intercal.
\end{aligned}
$$

Using the no arbitrage constraint (3) and idempotency of the projection matrix $\mathcal{P}_{\Sigma_{A,t}^\intercal}$, the PDE becomes

$$
\begin{aligned}
0 =& \partial_t h_t + h_t\left(\mu_{\hat{Q},t} - r_t - \Sigma_{\hat{Q},t}\Lambda_t\right) \\
& + (\partial_X h_t)^\intercal\left(\mu_{X,t} + \Sigma_{X,t}\left(\rho_{XA,t}\Sigma_{\hat{Q},t}^\intercal - \rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\Lambda_t\right)\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}h_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)
\end{aligned}
$$

Applying Itô's lemma to $h(t, X_t; T, \theta)$ shows that

$$
\mathrm{d}h_t = \partial_t h_t\,\mathrm{d}t + (\partial_X h_t)^\intercal\left(\mu_{X,t}\,\mathrm{d}t + \Sigma_{X,t}\,\mathrm{d}Z_{X,t}\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}h_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)\mathrm{d}t
$$

$h(t, X_t; T, \theta)$ is assumed to be both replicable and spanned by asset risk factors (Lemma 1), so combining (4) and (5) implies that

$$
(\partial_X h_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal} = (\partial_X h_t)^\intercal\Sigma_{X,t}\rho_{XA,t}.
$$

and replacing this term in the previous PDE yields

$$
\begin{aligned}
0 =& \partial_t h_t + h_t\left(\mu_{\hat{Q},t} - r_t - \Sigma_{\hat{Q},t}\Lambda_t\right) \\
& + (\partial_X h_t)^\intercal\left(\mu_{X,t} + \Sigma_{X,t}\left(\rho_{XA,t}\Sigma_{\hat{Q},t}^\intercal - \rho_{XA,t}\Lambda_t\right)\right) + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}h_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right).
\end{aligned}
$$

That PDE is equivalent to $\tilde{\Omega}(t, X_t; s)$ from PDE (8) for a payoff process $\hat{Q}_t$ with dynamics

$$
\frac{\mathrm{d}\hat{Q}_t}{\hat{Q}_t} = \mu_{\hat{Q},t}\,\mathrm{d}t + \Sigma_{\hat{Q},t}\,\mathrm{d}Z_{A,t}
$$

Applying a log-transformation and reformulating in terms of the mean-variance efficient portfolio $\tilde{A}_t$ process from Remark 2, dynamics simplify to

$$
\mathrm{d}\log\left(\hat{Q}_t\right) = -\frac{\delta_t\,\mathrm{d}t}{\gamma} + \frac{\mathrm{d}\log\left(\tilde{A}_t\right)}{\gamma} - \frac{1-\gamma}{\gamma}\theta^\intercal\,\mathrm{d}\log(P_t)
$$

which can be integrated as

$$
\hat{Q}_T = \hat{Q}_t\left(e^{-\int_t^T \delta_s \mathrm{d}s}\frac{\tilde{A}_T}{\tilde{A}_t}\right)^{\frac{1}{\gamma}} e^{\left(1-\frac{1}{\gamma}\right)\theta^\intercal(\log(P_T)-\log(P_t))}
$$

and reformulated in terms of $P_{\theta,t}^\star$

$$
\hat{Q}_T = \hat{Q}_t\left(e^{-\int_t^T \delta_s \mathrm{d}s}\frac{\tilde{A}_T}{\tilde{A}_t}\right)^{\frac{1}{\gamma}}\left(\frac{P_{\theta,T}^\star}{P_{\theta,t}^\star}\right)^{1-\frac{1}{\gamma}}.
$$

A.13 Proof of wealth under optimal policy, Remark 4

Wealth dynamics are given in (18)

$$\frac{\mathrm{d}W_t}{W_t} = (Q_t \mathbb{1}_{t\le T_R} - c_t^\star)\,\mathrm{d}t + W_t\left(r_t\,\mathrm{d}t + \pi_t^{\star\intercal}\left((\mu_{A,t} - r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\right)$$

and dynamics for human capital $\Upsilon(t, X_t, Q_t; T_R)$ are

$$\mathrm{d}\Upsilon_t = \partial_t\Upsilon_t\,\mathrm{d}t + (\partial_X\Upsilon_t)^\intercal\left(\mu_{X,t}\,\mathrm{d}t + \Sigma_{X,t}\,\mathrm{d}Z_{X,t}\right) + \partial_Q\Upsilon_t Q_t\left(\mu_{Q,t}\,\mathrm{d}t + \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t}\right).$$

After plugging the optimal consumption rate $c_t^\star$ from (26) and the optimal investment strategy $\pi_t^\star$ from (27), the dynamics of log total wealth at $t \in [t_0, T)$ become

$$\begin{aligned}
&\mathrm{d}\log(W_t + \Upsilon_t)\\
&= \frac{Q_t\mathbb{1}_{t\le T_R} + W_t r_t}{W_t + \Upsilon_t}\,\mathrm{d}t - \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde\theta,t}\right)^{1-\frac{1}{\gamma}}\frac{1}{f(t,X_t,P_t)}\,\mathrm{d}t\\
&\quad + \frac{(\mu_{A,t} - r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\left((\mu_{A,t} - r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\\
&\quad + \Bigg(\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\log(f_t)\\
&\qquad + \rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P\log(f_t)\\
&\qquad - \rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\frac{\partial_X\Upsilon_t}{W_t + \Upsilon_t}\\
&\qquad\qquad - \rho_{QA,t}^\intercal\Sigma_{Q,t}^\intercal\frac{Q_t\partial_Q\Upsilon_t}{W_t + \Upsilon_t}\Bigg)^\intercal\Sigma_{A,t}^\intercal\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\left((\mu_{A,t} - r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\\
&\quad + \frac{\partial_t\Upsilon_t\,\mathrm{d}t + (\partial_X\Upsilon_t)^\intercal\left(\mu_{X,t}\,\mathrm{d}t + \Sigma_{X,t}\,\mathrm{d}Z_{X,t}\right) + \partial_Q\Upsilon_t Q_t\left(\mu_{Q,t}\,\mathrm{d}t + \Sigma_{Q,t}\,\mathrm{d}Z_{Q,t}\right)}{W_t + \Upsilon_t}\\
&\quad + \frac{1}{2}\left(\frac{\operatorname{tr}\left(\partial_{XX^\intercal}\Upsilon_t\Sigma_{X,t}\Sigma_{X,t}^\intercal\right)}{W_t + \Upsilon_t} - \frac{(\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\Sigma_{X,t}^\intercal\partial_X\Upsilon_t}{(W_t + \Upsilon_t)^2}\right)\mathrm{d}t\\
&\quad + \frac{1}{2}\left(\frac{\operatorname{tr}\left(\partial_{QQ^\intercal}\Upsilon_t Q_t\Sigma_{Q,t}\Sigma_{Q,t}^\intercal Q_t\right)}{W_t + \Upsilon_t} - \frac{(\partial_Q\Upsilon_t)^2 Q_t^2\Sigma_{Q,t}\Sigma_{Q,t}^\intercal}{(W_t + \Upsilon_t)^2}\right)\mathrm{d}t\\
&\quad + \left(\frac{(\partial_{XQ}\Upsilon_t)^\intercal Q_t\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal}{W_t + \Upsilon_t} - \frac{(\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XQ,t}\Sigma_{Q,t}^\intercal\partial_Q\Upsilon_t Q_t}{(W_t + \Upsilon_t)^2}\right)\mathrm{d}t\\
&\quad - \frac{1}{2}\frac{(\mu_{A,t} - r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\frac{(\mu_{A,t} - r_t\mathbf{1})}{\gamma}\,\mathrm{d}t\\
&\quad - \frac{(\mu_{A,t} - r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma_{A,t}^\intercal\right)^{-1}\Sigma_{A,t}\left(\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\log(f_t) + \rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P\log(f_t)\right)\mathrm{d}t\\
&\quad - \frac{1}{2}(\partial_X\log(f_t))^\intercal\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\log(f_t)\,\mathrm{d}t\\
&\quad - \frac{1}{2}(\partial_P\log(f_t))^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{PA,t}^\intercal\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\partial_P\log(f_t)\,\mathrm{d}t\\
&\quad - (\partial_P\log(f_t))^\intercal\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\log(f_t)\,\mathrm{d}t\\
&\quad + \frac{1}{2}\frac{(\partial_X\Upsilon_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{XA,t}^\intercal\Sigma_{X,t}^\intercal\partial_X\Upsilon_t}{(W_t + \Upsilon_t)^2}\,\mathrm{d}t\\
&\quad + \frac{1}{2}\frac{\partial_Q\Upsilon_t Q_t\Sigma_{Q,t}\rho_{QA,t}\mathcal{P}_{\Sigma_{A,t}^\intercal}\rho_{QA,t}^\intercal\Sigma_{Q,t}^\intercal\partial_Q\Upsilon_t Q_t}{(W_t + \Upsilon_t)^2}\,\mathrm{d}t
\end{aligned}$$

$$+\frac{\partial_Q \Upsilon_t Q_t \Sigma_{Q,t}\rho_{QA,t}\mathcal{P}_{\Sigma^\intercal_{A,t}}\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X\Upsilon_t}{(W_t+\Upsilon_t)^2}\,\mathrm{d}t$$

Substituting $\partial_t\Upsilon_t$ from PDE (A.30) into dynamics for total wealth and invoking the assumption that human capital is tradeable yield

$$\begin{aligned}
&\mathrm{d}\log(W_t+\Upsilon_t)\\
=&r_t\,\mathrm{d}t-\varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{1-\frac{1}{\gamma}}\frac{1}{f(t,X_t,P_t)}\,\mathrm{d}t\\
&+\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}\left((\mu_{A,t}-r_t\mathbf{1})\,\mathrm{d}t+\Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\\
&-\frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}\frac{(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\,\mathrm{d}t\\
&+\Big(\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X\log(f_t)\\
&\qquad+\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\,\mathrm{diag}(P_t)\partial_P\log(f_t)\Big)^\intercal\Sigma^\intercal_{A,t}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}\left((\mu_{A,t}-r_t\mathbf{1})\,\mathrm{d}t+\Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)\\
&-\frac{1}{2}\Big(\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\,\mathrm{diag}(P_t)\partial_P\log(f_t)\\
&\qquad\qquad+\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X\log(f_t)\Big)\mathcal{P}_{\Sigma^\intercal_{A,t}}\Big(\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\,\mathrm{diag}(P_t)\partial_P\log(f_t)\\
&\qquad\qquad\qquad\qquad+\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X\log(f_t)\Big)\,\mathrm{d}t\\
&-\frac{(\mu_{A,t}-r_t\mathbf{1})^\intercal}{\gamma}\left(\Sigma_{A,t}\Sigma^\intercal_{A,t}\right)^{-1}\Sigma_{A,t}\left(\rho^\intercal_{XA,t}\Sigma^\intercal_{X,t}\partial_X\log(f_t)+\rho^\intercal_{PA,t}\Sigma^\intercal_{P,t}\,\mathrm{diag}(P_t)\partial_P\log(f_t)\right)\mathrm{d}t
\end{aligned}$$

Dynamics of $log(f_t)$ at $t\in[t_0,T)$ are

$$\begin{aligned}
\mathrm{d}log(f_t)=&\frac{\partial_t f_t}{f_t}\,\mathrm{d}t+\frac{(\partial_X f_t)^\intercal}{f_t}\,\mathrm{d}X_t+\frac{(\partial_P f_t)^\intercal}{f_t}\,\mathrm{d}P_t\\
&+\frac{1}{2}\,\mathrm{tr}\left(\left(\frac{\partial_{XX^\intercal}f_t}{f_t}-\frac{\partial_X f_t}{f_t}\frac{(\partial_X f_t)^\intercal}{f_t}\right)\mathrm{d}[X_t,X_t]\right)\\
&+\frac{1}{2}\,\mathrm{tr}\left(\left(\frac{\partial_{PP^\intercal}f_t}{f_t}-\frac{\partial_P f_t}{f_t}\frac{(\partial_P f_t)^\intercal}{f_t}\right)\mathrm{d}[P_t,P_t]\right)\\
&+\mathrm{tr}\left(\left(\frac{(\partial_{XP^\intercal}f_t)^\intercal}{f_t}-\frac{\partial_P f_t}{f_t}\frac{(\partial_X f_t)^\intercal}{f_t}\right)\mathrm{d}[X_t,P_t]\right).
\end{aligned}$$

Substitute $\partial_t f_t$ from PDE (A.19) and we arrive at

$$
\begin{aligned}
\mathrm{d}log(f_t) =& \Big(\rho_{XA,t}^{\intercal}\Sigma_{X,t}^{\intercal}\partial_X \log(f_t) \\
&\quad + \rho_{PA,t}^{\intercal}\Sigma_{P,t}^{\intercal}\operatorname{diag}(P_t)\partial_P \log(f_t)\Big)^{\intercal}\Sigma_{A,t}^{\intercal}\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}\left((\mu_{A,t}-r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right) \\
&+ \frac{(\partial_X f_t)^{\intercal}}{f_t}\left(\mathrm{d}X_t - \mu_{X,t}\,\mathrm{d}t - \Sigma_{X,t}\rho_{XA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\,\mathrm{d}Z_{A,t}\right) \\
&+ \frac{(\partial_P f_t)^{\intercal}}{f_t}\left(\mathrm{d}P_t - \operatorname{diag}(P_t)\mu_{P,t}\,\mathrm{d}t - \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\,\mathrm{d}Z_{A,t}\right) \\
&- \left(\frac{(\partial_X f_t)^{\intercal}}{f_t}\Sigma_{X,t}\rho_{XA,t} + \frac{(\partial_P f_t)^{\intercal}}{f_t}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\right)\Sigma_{A,t}^{\intercal}\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}\frac{(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\,\mathrm{d}t \\
&- \varepsilon_1^{\frac{1}{\gamma}}\left(P_{\tilde{\theta},t}^{\star}\right)^{1-\frac{1}{\gamma}}\frac{1}{f_t}\,\mathrm{d}t \\
&+ \frac{\delta_t - (1-\gamma)\left(r_t + \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^{\intercal}\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\right)}{\gamma}\,\mathrm{d}t \\
&- \frac{1}{2}\frac{(1-\gamma)}{f(t,X_t,P_t)^2}\Big((\partial_P f_t)^{\intercal}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t} \\
&\qquad\qquad + (\partial_X f_t)^{\intercal}\Sigma_{X,t}\rho_{XA,t}\Big)\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\Big((\partial_P f_t)^{\intercal}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t} \\
&\qquad\qquad\qquad + (\partial_X f_t)^{\intercal}\Sigma_{X,t}\rho_{XA,t}\Big)\,\mathrm{d}t \\
&- \gamma\Bigg(\frac{1}{2}\frac{(\partial_X f_t)^{\intercal}\,\mathrm{d}[X_t,X_t]\,\partial_X f_t}{f(t,X_t,P_t)^2} + \frac{1}{2}\frac{(\partial_P f_t)^{\intercal}\,\mathrm{d}[P_t,P_t]\,\partial_P f_t}{f(t,X_t,P_t)^2} \\
&\qquad\qquad + \frac{(\partial_X f_t)^{\intercal}\,\mathrm{d}[X_t,P_t]\,\partial_P f_t}{f(t,X_t,P_t)^2}\Bigg)\,\mathrm{d}t
\end{aligned}
$$

where, in relation to the first line, pairs of offsetting terms were strategically to make the next substitutions easier. Isolating those first line items and substituting them into the log total wealth dynamics, we arrive at

$$
\begin{aligned}
\mathrm{d}\log(W_t + \Upsilon_t) =& \mathrm{d}log(f_t) - \frac{\delta_t}{\gamma}\,\mathrm{d}t \\
&+ \frac{r_t\,\mathrm{d}t + (\mu_{A,t}-r_t\mathbf{1})^{\intercal}\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}\left((\mu_{A,t}-r_t\mathbf{1})\,\mathrm{d}t + \Sigma_{A,t}\,\mathrm{d}Z_{A,t}\right)}{\gamma} \\
&- \frac{1}{2}\frac{(\mu_{A,t}-r_t\mathbf{1})^{\intercal}\left(\Sigma_{A,t}\Sigma_{A,t}^{\intercal}\right)^{-1}(\mu_{A,t}-r_t\mathbf{1})}{\gamma}\,\mathrm{d}t \\
&- \left(\frac{(\partial_X f_t)^{\intercal}}{f_t}\Sigma_{X,t}\,\mathrm{d}Z_{X,t} + \frac{(\partial_P f_t)^{\intercal}}{f_t}\operatorname{diag}(P_t)\Sigma_{P,t}\,\mathrm{d}Z_{P,t}\right) \\
&+ \gamma\Bigg(\frac{1}{2}\frac{(\partial_X f_t)^{\intercal}\,\mathrm{d}[X_t,X_t]\,\partial_X f_t}{f(t,X_t,P_t)^2} + \frac{1}{2}\frac{(\partial_P f_t)^{\intercal}\,\mathrm{d}[P_t,P_t]\,\partial_P f_t}{f(t,X_t,P_t)^2} \\
&\qquad\qquad + \frac{(\partial_X f_t)^{\intercal}\,\mathrm{d}[X_t,P_t]\,\partial_P f_t}{f(t,X_t,P_t)^2}\Bigg)\,\mathrm{d}t \\
&+ \left(\frac{(\partial_X f_t)^{\intercal}}{f_t}\Sigma_{X,t}\rho_{XA,t} + \frac{(\partial_P f_t)^{\intercal}}{f_t}\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\right)\mathcal{P}_{\Sigma_{A,t}^{\intercal}}\,\mathrm{d}Z_{A,t}
\end{aligned}
$$

$$
\begin{aligned}
&- \gamma\frac{1}{2}\Big((\partial_P f_t)^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t} \\
&\qquad\qquad +(\partial_X f_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\Big)\frac{\mathcal{P}_{\Sigma^\intercal_{A,t}}}{f_t^2}\Big((\partial_P f_t)^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t} \\
&\qquad\qquad\qquad\qquad +(\partial_X f_t)^\intercal\Sigma_{X,t}\rho_{XA,t}\Big)\,\mathrm{d}t
\end{aligned}
$$

Using processes $\zeta_t, \tilde{\zeta}_t$ defined in (30) and $\tilde{A}_t$ from Remark 2, the dynamics of log total wealth simplify to

$$
\begin{aligned}
\mathrm{d}\log(W_t+\Upsilon_t) =& \mathrm{d}log(f_t) - \frac{\delta_t}{\gamma}\,\mathrm{d}t + \frac{1}{\gamma}\,\mathrm{d}\log\Big(\tilde{A}_t\Big) \\
&+ \frac{\mathrm{d}\tilde{\zeta}_t}{\tilde{\zeta}_t} - \gamma\frac{1}{2}\frac{\mathrm{d}[\tilde{\zeta}_t,\tilde{\zeta}_t]}{\tilde{\zeta}_t^2} - \left(\frac{\mathrm{d}\zeta_t}{\zeta_t} - \gamma\frac{1}{2}\frac{\mathrm{d}[\zeta_t,\zeta_t]}{\zeta_t^2}\right)
\end{aligned}
\tag{A.31}
$$

and more concisely

$$
\mathrm{d}\log(W_t+\Upsilon_t) = \mathrm{d}log(f_t) - \frac{\delta_t}{\gamma}\,\mathrm{d}t + \frac{1}{\gamma}\,\mathrm{d}\log\Big(\tilde{A}_t\Big) + \frac{1}{1-\gamma}\left(\frac{\mathrm{d}\tilde{\zeta}_t^{1-\gamma}}{\tilde{\zeta}_t^{1-\gamma}} - \frac{\mathrm{d}\zeta_t^{1-\gamma}}{\zeta_t^{1-\gamma}}\right)
$$

which integrates to

$$
\frac{W_t+\Upsilon_t}{f_t} = \left(\frac{W_{t_0}+\Upsilon_{t_0}}{f_{t_0}}\right)\left(e^{-\int_{t_0}^t \delta_s \mathrm{d}s}\frac{\tilde{A}_t}{\tilde{A}_{t_0}}\right)^{\frac{1}{\gamma}} \exp\left(\frac{1}{1-\gamma}\left(\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}\right)\right)
$$

The bundle consumption rate from Lemma 7 under the optimal consumption policy $c_t^\star$ from (26) becomes

$$
v_t^\star = v(c_t^\star, P_t, \tilde{\theta}) = \varepsilon_1^{\frac{1}{\gamma}}\left(P^\star_{\tilde{\theta},t}\right)^{-\frac{1}{\gamma}}\frac{W_{t_0}+\Upsilon_{t_0}}{f_{t_0}}\left(e^{-\int_{t_0}^t \delta_s \mathrm{d}s}\frac{\tilde{A}_t}{\tilde{A}_{t_0}}\right)^{\frac{1}{\gamma}} \exp\left(\frac{\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}}{1-\gamma}\right)
$$

implying that

$$
v_t^\star = v_{t_0}^\star\left(e^{-\int_{t_0}^t \delta_s \mathrm{d}s}\frac{P^\star_{\tilde{\theta},t_0}}{P^\star_{\tilde{\theta},t}}\frac{\tilde{A}_t}{\tilde{A}_{t_0}}\right)^{\frac{1}{\gamma}} \exp\left(\frac{\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}}{1-\gamma}\right).
$$

When $\gamma \to 1$ one can derive an equivalent expression starting from (A.31) where

$$
\exp\left(\frac{\int_{t_0}^t \frac{\mathrm{d}\tilde{\zeta}_s^{1-\gamma}}{\tilde{\zeta}_s^{1-\gamma}} - \int_{t_0}^t \frac{\mathrm{d}\zeta_s^{1-\gamma}}{\zeta_s^{1-\gamma}}}{1-\gamma}\right) \to \frac{\tilde{\zeta}_t}{\tilde{\zeta}_{t_0}}\frac{\zeta_{t_0}}{\zeta_t}.
$$

A.14 Proof of stationary lifecycle paths, Theorem 1

First I would like to draw attention towards the structure of problem (35). Parameters at time t depend only on contemporaneous age $t-t_0$ and state X_t. Probabilities of future states conditional on filtration depend only on current state X_t since the state process of this model is Markovian. The only extra problem variable to be tracked is current wealth to reference income ratio $\tilde{W}_t$. Therefore optimal policies only need to depend on age $t - t_0$, state X_t and $\tilde{W}_t$.

At t_0 indirect utility is

$$J(0, e^{\nu_{W_{t_0}}}, X_{t_0})$$

where the individual has age 0 and $\nu_{W_{t_0}}$ is the log of initial-wealth-to-reference-process ratio which only depends on X_{t_0}. From this point onwards, lifepath processes (37) can be characterized as pure mapping functions of current age $t - t_0$ and of realized paths comprising the state X_t path and the paths of increments followed by $Z_{A,t}$ and $Z_{Y,t}$ since t_0

$$\left\{X_{[t_0,t]}, \Delta Z_{A,[t_0,t]}, \Delta Z_{Y,[t_0,t]}\right\}. \tag{A.32}$$

The pushforward distributions of paths for lifecycle processes (37) are mappings over the distribution of state paths, asset return increment paths and reference process increment paths (A.32). When these state and increment processes are stationary (36), then by Kallenberg (2021, Lemma 25.1) the distributions of lifecycle paths associated to processes (37) are also stationary.

This characterization holds also under relaxed assumptions allowing parameters τ_{T,t_0}, τ_{R,t_0}, γ, θ, $\tilde{\theta}$, ε_1, ε_2 to depend on X_{t_0}. These parameters are already resolved at t_0 when the individual starts making decisions, so the problem conditional on these parameters is still equivalent to (35).

It even holds when modelling $\tau_{T,t}$, $\tau_{R,t}$ as random stopping times given by the first arrival since t_0 of their respective Poisson process $N_{T,t}$ and $N_{R,t}$ with non-negative hazard rate process $\lambda_{T,t} \coloneqq \lambda_T(t - t_0, X_t)$ and $\lambda_{R,t} \coloneqq \lambda_R(t - t_0, X_t)$ dependent on age $t - t_0$ and state X_t. In this case, the terminal and retirement stopping times become

$$\begin{aligned}\tau_{T,t} &= \inf\{m | m > t_0 \text{ and } N_{T,m} - N_{T,t_0} \geq 1\} \\ \tau_{R,t} &= \inf\{m | m > t_0 \text{ and } (N_{R,m} - N_{R,t_0} \geq 1 \text{ or } N_{T,m} - N_{T,t_0} \geq 1)\}.\end{aligned}$$

Compared to problem (35), the state space includes now extra information related to these jumps, i.e. $J(t - t_0, \tilde{W}_t, X_t, N_{T,t} - N_{T,t_0}, N_{R,t} - N_{R,t_0})$. Lifepath processes (37) can be still characterized as pure mapping functions of current age $t - t_0$ and the following realized paths

$$\left\{X_{[t_0,t]}, \Delta Z_{A,[t_0,t]}, \Delta Z_{Y,[t_0,t]}, \Delta N_{T,[t_0,t]}, \Delta N_{R,[t_0,t]}\right\}. \tag{A.33}$$

Since the intensity of jump processes $N_{T,t}, N_{R,t}$ depends only on age $t - t_0$ and state X_t, then stationarity in (A.32) implies that (A.33) is also stationary.

A.15 Proof of deterministic lifecycle processes under extreme risk aversion, Corollary 2

First, $\frac{Q_t}{Y_t} = e^{\nu_{Q,t}}$ becomes an age $t - t_0$ dependent function after removing the state X_t. $\frac{P^\star_{\theta,t}}{Y_t}$, $\frac{P^\star_{\tilde{\theta},t}}{Y_t}$ become constant when there is no state process X_t and $\frac{c^\star_t}{P^\star_{\tilde{\theta},t}} = v(c^\star_t, P_t, \tilde{\theta})$ is already shown to be constant in Remark 5. From this, it follows that $\tilde{c}^\star_t = \frac{c^\star_t}{P^\star_{\tilde{\theta},t}} \frac{P^\star_{\tilde{\theta},t}}{Y_t}$ is constant as well.

Under stated assumptions and with time horizon $\tau_{R,t} = T_R - t$, PDE (8) characterizes a price multiplier $\bar{\Omega}(t - t_0; \tau_{R,t})$ with boundary condition $\bar{\Omega}(T_R - t_0; 0) = 1$ that only depends on age $t - t_0$ but not directly depend on time or X_t. Notice that the time horizon $\tau_{T,t}$ is a deterministic function of age $t - t_0$ in this setup. Then from Lemma 4, the right hand side of the following equation depends only on age $t - t_0$

$$\frac{\Upsilon(t, X_t, Q_t; T_R)}{Y_t} = \frac{Q_t}{Y_t} \int_0^{\tau_{R,t}} \tilde{\Omega}(t, X_t; t+s) \, \mathrm{d}s = e^{\nu_{Q,t}} \int_0^{\tau_{R,t}} \bar{\Omega}(t - t_0; s) \, \mathrm{d}s$$

Under stated assumptions and with time horizon $\tau_{T,t}$, PDE (A.21) characterizes a price multiplier $\bar{h}(t - t_0; \tau_{T,t})$ with boundary condition $\bar{h}(T - t_0; 0) = 1$ that only depends on age $t - t_0$ but not directly depend on time or X_t. Notice that the time horizon $\tau_{T,t}$ is a deterministic function of age $t - t_0$ in this setup. Then simplifying (28) using assumptions and previous arguments, the right hand side of the following equation depends only on age $t - t_0$

$$\begin{aligned}
\frac{f(t, X_t, P_t)}{Y_t} =& \frac{P^\star_{\theta,t}}{Y_t} \left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t, X_t; t + \tau_{T,t}, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_0^{\tau_{T,t}} h(t, X_t; t+s, \theta) \, \mathrm{d}s \right) \\
=& \frac{P^\star_{\theta,t}}{Y_t} \left(\mathbb{1}_{\varepsilon_2 \neq 0} \bar{h}(t - t_0; \tau_{T,t}, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_0^{\tau_{T,t}} \bar{h}(t - t_0; s, \theta) \, \mathrm{d}s \right)
\end{aligned}$$

From Remark 5

$$\frac{\tilde{W}_t + \frac{\Upsilon(t, X_t, Q_t; T_R)}{Y_t}}{\frac{f(t, X_t, P_t)}{Y_t}} = \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)} = \frac{W_{t_0} + \Upsilon(t_0, X_{t_0}, Q_{t_0}; T_R)}{f(t_0, X_{t_0}, P_{t_0})}$$

so $\tilde{W}_t$ is a composition of the purely time dependent functions described before

$$\tilde{W}_t = \frac{W_{t_0} + \Upsilon(t_0, X_{t_0}, Q_{t_0}; T_R)}{f(t_0, X_{t_0}, P_{t_0})} \frac{f(t, X_t, P_t)}{Y_t} - \frac{\Upsilon(t, X_t, Q_t; T_R)}{Y_t}.$$

The process J_t from (35) becomes deterministic since there is no state X_t and its remaining inputs are deterministic $J(t - t_0, \tilde{W}_t)$.

Optimal investment strategy $\pi^\star_t$ from (27) becomes constant

$$\begin{aligned}
\pi^\star_t =& \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) \partial_P \log(f_t) \frac{\tilde{W}_t + \frac{\Upsilon_t}{Y_t}}{\tilde{W}_t} - \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{QA}^\intercal \Sigma_Q^\intercal \frac{\frac{\Upsilon_t}{Y_t}}{\tilde{W}_t} \\
=& \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{YA}^\intercal \Sigma_Y^\intercal \frac{\tilde{W}_t + \frac{\Upsilon_t}{Y_t}}{\tilde{W}_t} - \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{YA}^\intercal \Sigma_Y^\intercal \frac{\frac{\Upsilon_t}{Y_t}}{\tilde{W}_t} \\
=& \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{YA}^\intercal \Sigma_Y^\intercal.
\end{aligned}$$

A.16 Proof of equivalent initial wealth, Lemma 8

Because of strict monotonicity on initial wealth W_t, expected utility can replace initial wealth in the minimization objective (33)

$$\begin{aligned}
\mathrm{EW}(t, W_t, H_t; \pi, \bar{\pi}) = \min_{\tilde{W}} \; & U(t, \tilde{W}, H_t; \bar{\pi}) \\
\text{s.t. } & U(t, \tilde{W}, H_t; \bar{\pi}) \geq U(t, W_t, H_t; \pi).
\end{aligned}$$

The range of U does not depend on the strategy and the function is continuous on initial wealth, so there exists a $\tilde{W}$ such that $U(t,\tilde{W},H_t;\bar{\pi}) = U(t,W_t,H_t;\pi)$. This value coincides with the minimum to the problem above and strict monotonicity makes the function invertible, thus the equivalent initial wealth reduces to

$$\mathrm{EW}(t,W_t,H_t;\pi,\bar{\pi}) = \left(U(t,\cdot,H_t;\bar{\pi})\right)^{-1}\left(U(t,W_t,H_t;\pi)\right).$$

A.17 Proof of expected utility with incomplete markets, Lemma 9

The Hamilton-Jacobi-Bellman equation can be derived following the steps of Section A.10 up to (A.14). With $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $Q_t = 0$, $c_t = 0$ and for any arbitrary but finite π, we arrive at

$$\begin{aligned}
0 =& \partial_W U_t W_t \pi_t^\intercal(\mu_{A,t} - r_t\mathbf{1}) \\
&+ \frac{1}{2}\partial_{WW}U_t W_t^2 \pi_t^\intercal \Sigma_{A,t}\Sigma_{A,t}^\intercal \pi_t + (\partial_{WX}U_t)^\intercal \Sigma_{X,t}\rho_{XA,t}\Sigma_{A,t}^\intercal \pi_t W_t \\
&+ (\partial_{WP}U_t)^\intercal \operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal \pi_t W_t \\
&- \delta_t U(t,W_t,X_t,P_t;\pi) + \partial_t U_t + \partial_W U_t W_t r_t + (\partial_X U_t)^\intercal \mu_{X,t} + \frac{1}{2}\operatorname{tr}\left(\partial_{XX^\intercal}U_t \Sigma_{X,t}\Sigma_{X,t}^\intercal\right) \\
&+ (\partial_P U_t)^\intercal \operatorname{diag}(P_t)\mu_{P,t} + \frac{1}{2}\operatorname{tr}\left(\partial_{PP^\intercal}U_t \operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\right) \\
&+ \operatorname{tr}\left((\partial_{XP^\intercal}U_t)^\intercal \Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal \operatorname{diag}(P_t)\right)
\end{aligned}$$

with boundary condition $U(T,W_t,X_t,P_t;\pi) = \left(P_{\theta,t}^\star\right)^{\gamma-1}\frac{W_T^{1-\gamma}}{1-\gamma}$.

Consider the following ansatz

$$U(t,W_t,X_t,P_t;\pi) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t,X_t,P_t;\pi)^\gamma$$

then

$$\begin{aligned}
0 =& \partial_t f_t \\
&- f(t,X_t,P_t;\pi)\frac{\delta_t - (1-\gamma)\left(r_t + \pi_t^\intercal(\mu_{A,t} - r_t\mathbf{1}) - \gamma\frac{1}{2}\pi_t^\intercal\Sigma_{A,t}\Sigma_{A,t}^\intercal\pi_t\right)}{\gamma} \\
&+ (\partial_X f_t)^\intercal\left(\mu_{X,t} + (1-\gamma)\Sigma_{X,t}\rho_{XA,t}\Sigma_{A,t}^\intercal\pi_t\right) \\
&+ f(t,X_t,P_t;\pi)\frac{1}{2}\operatorname{tr}\left(\left((\gamma-1)f(t,X_t,P_t;\pi)^{-2}\partial_X f_t(\partial_X f_t)^\intercal + f(t,X_t,P_t;\pi)^{-1}\partial_{XX^\intercal}f_t\right)\Sigma_{X,t}\Sigma_{X,t}^\intercal\right) \\
&+ (\partial_P f_t)^\intercal\left(\operatorname{diag}(P_t)\mu_{P,t} + (1-\gamma)\operatorname{diag}(P_t)\Sigma_{P,t}\rho_{PA,t}\Sigma_{A,t}^\intercal\pi_t\right) \\
&+ \frac{1}{2}\operatorname{tr}\left(\left((\gamma-1)f(t,X_t,P_t;\pi)^{-1}\partial_P f_t(\partial_P f_t)^\intercal + \partial_{PP^\intercal}f_t\right)\operatorname{diag}(P_t)\Sigma_{P,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right) \\
&+ \operatorname{tr}\left(\left((\gamma-1)f(t,X_t,P_t;\pi)^{-1}\partial_X f_t(\partial_P f_t)^\intercal + \partial_{XP^\intercal}f_t\right)^\intercal\Sigma_{X,t}\rho_{XP,t}\Sigma_{P,t}^\intercal\operatorname{diag}(P_t)\right)
\end{aligned}$$

with boundary condition $f(T,X_t,P_t;\pi) = \left(P_{\theta,t}^\star\right)^{1-\frac{1}{\gamma}}$.

Using the following ansatz

$$f(t,X_t,P_t;\pi) = \left(P_{\theta,t}^\star\right)^{1-\frac{1}{\gamma}} h(t,X_t;T,\theta,\pi)$$

then

$$
\begin{aligned}
\partial_t h_t =& h(t, X_t; T, \theta, \pi) \Bigg(\frac{\delta_t}{\gamma} - \left(\frac{1}{\gamma} - 1\right) \Bigg(r_t + (\mu_{A,t} - r_t \mathbf{1})^\intercal \pi_t - \frac{1}{2} \gamma \pi_t^\intercal \Sigma_{A,t} \Sigma_{A,t}^\intercal \pi_t \\
& \qquad - \theta^\intercal \left(\mu_{P,t} + (1-\gamma) \Sigma_{P,t} \rho_{PA,t} \Sigma_{A,t}^\intercal \pi_t \right) \\
& \qquad + \frac{1}{2}(1-\gamma) \theta^\intercal \Sigma_{P,t} \Sigma_{P,t}^\intercal \theta + \frac{1}{2} \operatorname{tr} \left(\operatorname{diag}(\theta) \Sigma_{P,t} \Sigma_{P,t}^\intercal \right) \Bigg) \Bigg) \\
& - (\partial_X h_t)^\intercal \left(\mu_{X,t} + (1-\gamma) \Sigma_{X,t} \left(\rho_{XA,t} \Sigma_{A,t}^\intercal \pi_t - \rho_{XP,t} \Sigma_{P,t}^\intercal \theta \right) \right) \\
& + \frac{1}{2}(1-\gamma) \frac{(\partial_X h_t)^\intercal \Sigma_{X,t} \Sigma_{X,t}^\intercal \partial_X h_t}{h(t, X_t; T, \theta, \pi)} - \frac{1}{2} \operatorname{tr} \left(\partial_{XX^\intercal} h_t \Sigma_{X,t} \Sigma_{X,t}^\intercal \right)
\end{aligned} \tag{A.34}
$$

with boundary condition $h(T, X_t; T, \theta, \pi) = 1$.

The PDE for $h(t, X_t; T, \theta, \pi)$ is a particular case of the semi-linear PDE from Definition 3 parametrized as

$$
\begin{aligned}
g\Big(& t, X_t; \\
& R_t = -\frac{\delta_t}{\gamma} + \left(\frac{1}{\gamma} - 1\right) \Big(r_t + (\mu_{A,t} - r_t \mathbf{1})^\intercal \pi_t - \frac{1}{2} \gamma \pi_t^\intercal \Sigma_{A,t} \Sigma_{A,t}^\intercal \pi_t \\
& \qquad - \theta^\intercal \left(\mu_{P,t} + (1-\gamma) \Sigma_{P,t} \rho_{PA,t} \Sigma_{A,t}^\intercal \pi_t \right) \\
& \qquad + \frac{1}{2}(1-\gamma) \theta^\intercal \Sigma_{P,t} \Sigma_{P,t}^\intercal \theta + \frac{1}{2} \operatorname{tr} \left(\operatorname{diag}(\theta) \Sigma_{P,t} \Sigma_{P,t}^\intercal \right) \Big), \\
& B_t = \mu_{X,t} + (1-\gamma) \Sigma_{X,t} \left(\rho_{XA,t} \Sigma_{A,t}^\intercal \pi_t - \rho_{XP,t} \Sigma_{P,t}^\intercal \theta \right), \\
& C_t = \gamma \Sigma_{X,t} \Sigma_{X,t}^\intercal, \\
& D_t = \Sigma_{X,t} \Sigma_{X,t}^\intercal \Big)
\end{aligned} \tag{A.35}
$$

with boundary condition $g(T, X_t) = 1$.

Section A.21 explains how to reduce the semi-linear PDE to a system of Riccati ODEs when parameters satisfy a quadratic structure in state. In that case, R_t, B_t, C_t, D_t can be constructed from the following building blocks

$$
\begin{aligned}
(\delta_t) &= {}_{\delta}\alpha + {}_{\delta}\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_{\delta}{\omega^h}_m \, {\eta^m}_q \, X^q \\
(r_t) &= {}_{r}\alpha + {}_{r}\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_{r}{\omega^h}_m \, {\eta^m}_q \, X^q \\
((\mu_{A,t} - r_t \mathbf{1})^\intercal \pi_t) &= {}_{A}\alpha + {}_{A}\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_{A}{\omega^h}_m \, {\eta^m}_q \, X^q \\
(\pi_t^\intercal \Sigma_{A,t} \Sigma_{A,t}^\intercal \pi_t) &= {}_{\Sigma_A}\alpha + {}_{\Sigma_A}\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_A}{\omega^h}_m \, {\eta^m}_q \, X^q \\
(\mu_{P,t})^k &= {}_{P}\alpha^k + {}_{P}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{P}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
\left(\Sigma_{P,t} \rho_{PA,t} \Sigma_{A,t}^\intercal \pi_t\right)^k &= {}_{\Sigma_P A}\alpha^k + {}_{\Sigma_P A}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_P A}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
{\left(\Sigma_{P,t} \Sigma_{P,t}^\intercal\right)^k}_l &= {}_{\Sigma_P}{\alpha^k}_l + {}_{\Sigma_P}{\beta^k}_{lp} \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_P}{\omega^k}_l{}^h{}_m \, {\eta^m}_q \, X^q \\
(\mu_{X,t})^k &= {}_{X}\alpha^k - {}_{X}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{X}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
\left(\Sigma_{X,t} \rho_{XA,t} \Sigma_{A,t}^\intercal \pi_t\right)^k &= {}_{\Sigma_X A}\alpha^k + {}_{\Sigma_X A}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X A}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
{\left(\Sigma_{X,t} \rho_{XP,t} \Sigma_{P,t}^\intercal\right)^k}_l &= {}_{\Sigma_X \Sigma_P}{\alpha^k}_l + {}_{\Sigma_X \Sigma_P}{\beta^k}_{lp} \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X \Sigma_P}{\omega^k}_l{}^h{}_m \, {\eta^m}_q \, X^q \\
{\left(\Sigma_{X,t} \Sigma_{X,t}^\intercal\right)^k}_l &= {}_{\Sigma_X}{\alpha^k}_l + {}_{\Sigma_X}{\beta^k}_{lp} \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X}{\omega^k}_l{}^h{}_m \, {\eta^m}_q \, X^q
\end{aligned} \tag{A.36}
$$

Section A.22 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases. This explicit solution assumes time invariant coefficients but, when the investment strategy π depends on time, Corollary 3 can help to isolate time dependent coefficents in a separate PDE and leave only time invariant coefficients for Section A.22.

A.18 Proof of semi-linear PDE separation, Lemma 10

Using the Ansatz

$$g(t,X) = g_1(t,X_{1,t})g_2(t,X_{2,t})$$

and substituting into (38) we arrive at

$$\begin{aligned}0 =& g_2(t,X_{2,t})\Bigg(\partial_t g_{1,t} + (\partial_{X_1} g_{1,t})^\intercal \left(B_{1,t} + \tilde{C}_{1,t}\partial_{X_2}\log g_{2,t}\right) + \frac{1}{2}\frac{(\partial_{X_1} g_{1,t})^\intercal (C_{1,t} - D_{1,t})\partial_{X_1} g_{1,t}}{g_1(t,X_{1,t})} \\ &+ \frac{1}{2}\operatorname{tr}\left(D_{1,t}\partial_{X_1X_1^\intercal} g_{1,t}\right) + g_1(t,X_{1,t})\Bigg(R_{1,t} + (\partial_{X_2}\log g_{2,t})^\intercal \hat{B}_{1,t} \\ &+ \frac{1}{2}(\partial_{X_2}\log g_{2,t})^\intercal \hat{C}_{1,t}\partial_{X_2}\log g_{2,t} \\ &+ \frac{1}{2}\operatorname{tr}\left(\hat{D}_{1,t}\partial_{X_2X_2^\intercal}\log g_{2,t}\right)\Bigg)\Bigg) \\ &+ g_1(t,X_{1,t})\Bigg(\partial_t g_{2,t} + g_2(t,X_{2,t})R_{2,t} + (\partial_{X_2} g_{2,t})^\intercal B_{2,t} + \frac{1}{2}\frac{(\partial_{X_2} g_{2,t})^\intercal (C_{2,t} - D_{2,t})\partial_{X_2} g_{2,t}}{g_2(t,X_{2,t})} \\ &+ \frac{1}{2}\operatorname{tr}\left(D_{2,t}\partial_{X_2X_2^\intercal} g_{2,t}\right)\Bigg)\end{aligned}$$

which is satisfied when PDEs (41) (42) hold. Under conditions 1, 2, the parameters of PDE (42) only dependent on $X_{1,t}$ or time, but not on $X_{2,t}$. Parameters of PDE (41) only depend on $X_{2,t}$ or time.

A.19 Proof of time separation, Corollary 3

Applying Lemma 10 with an empty state partition $X_{1,t}$ reduces the PDE (42) to an ODE for $g_1(t)$

$$\begin{aligned}0 =& \partial_t g_{1,t} + g_1(t)\Bigg(R_{1,t} + (\partial_{X_2}\log g_{2,t})^\intercal \hat{B}_{1,t} + \frac{1}{2}(\partial_{X_2}\log g_{2,t})^\intercal \hat{C}_{1,t}\partial_{X_2}\log g_{2,t} \\ &+ \frac{1}{2}\operatorname{tr}\left(\hat{D}_{1,t}\partial_{X_2X_2^\intercal}\log g_{2,t}\right)\Bigg)\end{aligned}$$

and the solution to this linear ODE with boundary condition $g_1(T)$ is just

$$g_1(t) = g_1(T)\exp\left(\int_t^T \left(R_{1,s} + (\partial_{X_2}\log g_{2,s})^\intercal \hat{B}_{1,s} + \frac{1}{2}(\partial_{X_2}\log g_{2,s})^\intercal \hat{C}_{1,s}\partial_{X_2}\log g_{2,s} + \frac{1}{2}\operatorname{tr}\left(\hat{D}_{1,s}\partial_{X_2X_2^\intercal}\log g_{2,s}\right)\right)\mathrm{d}s\right).$$

A.20 Brief explanation of tensor notation

A tensor is a generalization of vectors and matrices that can accommodate any arbitrary number of axes.

Consider matrices A and B of size $n \times n$, column vectors x, y and z of size n and a scalar c. Using tensor notation we need refer to the matrix $A^i{}_j$ specifying the indices i for the row axis and j for the column axis. The position of indices determines the axes that they refer to, for instance $A^j{}_i$ refers to rows with j and to columns with i but it is otherwise equivalent to our previous example except for the change in "labels". If an index is omitted, it is understood that we contract the omitted axis and sum its values. E.g. writing A^i in tensor notation would be equivalent to $A\mathbf{1}$ in matrix notation.

Indices need to be specified in both sides of an assignment, and they can appear as subscripts (covariant) or superscripts (contravariant). When the same index appears twice in a product expression once as a subscript and once as a superscript regardless of product order, it is equivalent to the inner product, that is

$$y^i \;=\; A^i{}_j\, x^j = x^j\, A^i{}_j \;=\; A^{ij}\, x_j = x_j\, A^{ij}$$

is equivalent to $y = Ax$ in matrix notation. If an index appears twice as a superscript or subscript on different tensors regardless of product order, then it is equivalent to an elementwise product, e.g. $z^i = x^i y^i = y^i x^i$ is equivalent to $z = \operatorname{diag}(x)\, y$. The outer product $A = xy^\intercal$ in matrix notation corresponds to $A^i{}_j \;=\; x^i\, y_j \;=\; y_j\, x^i$ in tensor notation. If an unassigned index appears twice on the same tensor, once as covariant and once as contravariant $c = A^i{}_i$, it is equivalent to the trace in matrix notation $c = \operatorname{tr}(A)$. An assigned index appearing twice as covariant or contravariant on the same tensor $x^i \;=\; A^{ii}$ extracts its diagonal $x = \operatorname{diag}(A)$. With these rules it is also easy to see that the matrix transpose $B = A^\intercal$ is equivalent to $B^i{}_j \;=\; A_j{}^i$.

In this environment it is handy to define the Kronecker delta δ

$$\delta^i{}_j \;= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \tag{A.37}$$

then building a diagonal matrix from a vector $A = \operatorname{diag}(x)$ becomes $A^i{}_j \;=\; x^i\, \delta^i{}_j$.

Tensor notation is very flexible and allows some expressions that cannot be translated to matrix notation. While in matrices the first axis is contravariant and the second axis is covariant, tensor notation allows the expression A_{ij} making both indices covariant. The rules explained here extend quite naturally to tensors of more than 2 axes, like $C^i{}_{jk}{}^l$.

Subscripts and superscripts on the main tensor symbol are part of the tensor object. For instance descriptive subscripts C, t and D, t add a time index τ and help to distinguish tensors ${}_{C,\tau}\beta^k{}_{lp}$ and ${}_{D,\tau}\beta^k{}_{lp}$ that have axes k, l, p.

A.21 Reducing semi-linear PDE to Riccati ODEs, Lemma 11

Consider the PDE (38) for $g(t, X_t)$ with some deterministic boundary condition $g(T, X_t)$ and dynamic parameters $R_t := R(t, X_t)$ as a scalar, $B_t := B(t, X_t)$ of size n_X, $C_t := C(t, X_t)$ symmetric of size $n_X \times n_X$ and $D_t := D(t, X_t)$ symmetric of size $n_X \times n_X$

$$0 =\partial_t g_t + g_t R_t + (\partial_X g_t)^\intercal B_t + \frac{1}{2}\frac{(\partial_X g_t)^\intercal (C_t - D_t)\, \partial_X g_t}{g(t, X_t)} + \frac{1}{2}\operatorname{tr}(\partial_{XX^\intercal} g_t D_t).$$

This section uses tensor notation described in Section A.20. Assume that the boundary condition can be decomposed in terms of the following deterministic coefficients

$$g(T, X) =e^{a(0)+X_i\, b(0)^i + \frac{1}{2} X_i\, {\eta_\mu}^i\, c(0)^\mu{}_\nu\, \eta^\nu{}_j\, X^j}.$$

Reducing the PDE (38) into Riccati ODEs is possible when state dependence can be reparametrized using quadratic terms as in Liu (2007). This can be expressed in tensor notation and using time horizon $\tau = T - t$ as

$$\begin{aligned}
(R_{T-\tau}) &= {}_{R,\tau}\alpha + {}_{R,\tau}\beta_p\, X^p + X_p\, {\eta_h}^p\, {}_{R,\tau}\omega^h{}_m\, \eta^m{}_q\, X^q \\
(B_{T-\tau})^k &= {}_{B,\tau}\alpha^k + {}_{B,\tau}\beta^k{}_p\, X^p + X_p\, {\eta_h}^p\, {}_{B,\tau}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
(C_{T-\tau})^k{}_l &= {}_{C,\tau}\alpha^k{}_l + {}_{C,\tau}\beta^k{}_{lp}\, X^p + X_p\, {\eta_h}^p\, {}_{C,\tau}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
(D_{T-\tau})^k{}_l &= {}_{D,\tau}\alpha^k{}_l + {}_{D,\tau}\beta^k{}_{lp}\, X^p + X_p\, {\eta_h}^p\, {}_{D,\tau}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q
\end{aligned}$$

where parameters subscripted with left subscripts ${}_C$ and ${}_D$ are symmetric with respect to the first two indices, e.g. ${}_{C,\tau}\beta^k{}_{lp} = {}_{C,\tau}\beta_l{}^k{}_p$. On the right hand side, $\eta^m{}_q$ is constant and remaining parameters ${}_{\square,\tau}\alpha$, ${}_{\square,\tau}\beta$, ${}_{\square,\tau}\omega$ can still be time dependent through time horizon τ but not state X dependent. Time subindex on X is omitted to reduce notation clutter.

This general formulation includes those of Kim and Omberg (1996), Wachter (2002) and Liu (2007). The solution to the PDE is based upon the following ansatz

$$g(t, X) =e^{a(T-t)+X_i\, b(T-t)^i + \frac{1}{2} X_i\, {\eta_\mu}^i\, c(T-t)^\mu{}_\nu\, \eta^\nu{}_j\, X^j}.$$

Main Riccati ODEs Using $\tau = T - t$ and substituting the ansatz into the PDE (38) yields, under restrictions (A.41)-(A.43) and (A.44) detailed below, the following system of coupled

ODEs[3]

$$
\partial_\tau a = {}_{R,\tau}\alpha + {}_{B,\tau}\alpha^k\, b(\tau)_k + \frac{1}{2}\, {}_{C,\tau}\alpha^k{}_l\, b(\tau)_k\, b(\tau)^l + \frac{1}{2}\, {}_{D,\tau}\alpha^k{}_l\, \eta_\mu{}^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} \tag{A.38}
$$

$$
\begin{aligned}
\partial_\tau b^i =& {}_{R,\tau}\beta^i + {}_{B,\tau}\beta^{ki}\, b(\tau)_k + \frac{1}{2}\, {}_{C,\tau}\beta^k{}_l{}^i\, b(\tau)_k\, b(\tau)^l + {}_{C,\tau}\alpha^k{}_l\, \eta^\nu{}_k\, \eta_\mu{}^i\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2}\, b(\tau)^l \\
&+ \left({}_{B,\tau}\alpha^k\, \eta_\mu{}^i + \frac{1}{2}\, {}_{D,\tau}\beta^k{}_l{}^i\, \eta_\mu{}^l \right) \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2}
\end{aligned} \tag{A.39}
$$

$$
\begin{aligned}
\partial_\tau c^i{}_j =& 2\, {}_{R,\tau}\omega^i{}_j + 2\, {}_{B,\tau}\hat{\beta}_j{}^\nu\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2} + {}_{D,\tau}\omega^k{}_l{}^i{}_j\, \eta_\mu{}^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} \\
&+ {}_{C,\tau}\alpha^k{}_l\, \eta^\nu{}_k\, \eta_\mu{}^l\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2}\, \frac{c(\tau)^\mu{}_j + c(\tau)_j{}^\mu}{2} + 2\, {}_{B,\tau}\omega^{ki}{}_j\, b(\tau)_k + {}_{C,\tau}\omega^k{}_l{}^i{}_j\, b(\tau)^l\, b(\tau)_k
\end{aligned} \tag{A.40}
$$

with the previously assumed $a(0)$, $b(0)^i$, $c(0)^i{}_j$ as boundary conditions.

The following restrictions remove the cubic and quartic X terms from the PDE

$$
{}_{B,\tau}\omega^{kh}{}_m\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} = 0 \tag{A.41}
$$

$$
{}_{C,\tau}\omega^k{}_l{}^h{}_m\, b(\tau)^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} = 0 \tag{A.42}
$$

$$
{}_{C,\tau}\beta^k{}_l{}^p\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} = 0 \tag{A.43}
$$

while additionally imposing this other restriction helps to simplify the ODE for $c^i{}_j$ in (A.40)

$$
{}_{B,\tau}\beta^k{}_q\, \eta^\nu{}_k\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2} = {}_{B,\tau}\hat{\beta}_j{}^\nu\, \eta^j{}_q\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2} \tag{A.44}
$$

where ${}_{B,\tau}\hat{\beta}_j{}^\nu$ is defined implicitly through this restriction.

In general the above system of Riccati equations is not guaranteed to have a unique solution and even existence of a solution can depend on the parameters and time horizon considered. The numerical procedure to solve for $a(\tau)$ is straightforward: integrate (A.38) using initial condition $a(0)$. Numerically solving (A.39) and (A.40) for $b(\tau)^i$ and $c(\tau)^i{}_j$ is also possible, e.g. discretizing, but it is more computationally involved. If parameters are time varying, one also needs to apply the appropriate transformation so that they are indexable by $\tau = T - t$.

A solution to a simple case In the special case of ${}_{R,\tau}\beta^i = 0$, ${}_{R,\tau}\omega^i{}_j = 0$, $b(0)^i = 0$ and $c(0)^i{}_j = 0$, the restrictions (A.41)-(A.43) and (A.44) are not needed, since those terms involve $c(\tau)^i{}_j$ and its value is zero. The solution in this case is simply

$$
\begin{aligned}
a(\tau) =& a(0) + \int_0^\tau {}_{R,s}\alpha \; \mathrm{d}s \\
b(\tau)^i =& 0 \\
c(\tau)^i{}_j =& 0.
\end{aligned}
$$

[3]The ODE for $c^i{}_j$ admits an alternative definition to (A.40), where the term ${}_{B,\tau}\hat{\beta}_j{}^\nu\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2}$ is replaced by ${}_{B,\tau}\hat{\beta}^i{}_\mu\, \frac{c(\tau)^\mu{}_j + c(\tau)_j{}^\mu}{2}$. The definition of choice is simply a matter of convetion and both produce the same final expression $g(t, X_t)$ because the quadratic term is symmetric. Some terms related to C were also simplified based on this symmetry.

Diagonalized Riccati ODEs Some interesting closed form solutions can be obtained by diagonalization when parameters are constant. Let δ denote the Kronecker delta (A.37) and the binary operator $\odot$ denote the Hadamard product. The Riccati ODEs can be diagonalized with $c(\tau) = \operatorname{diag}(\tilde{c}_1(\tau), \tilde{c}_2(\tau), \ldots)$ to partly decouple the ODEs as

$$\begin{aligned}
\partial_\tau a =& \upsilon_0 + \upsilon_3^\intercal b(\tau) + \frac{1}{2}\operatorname{tr}(\ell_5 b(\tau) b(\tau)^\intercal) + \frac{1}{2}\operatorname{tr}(\ell_7 \operatorname{diag}(\tilde{c}(\tau))) \\
\partial_\tau b_i =& \upsilon_{1,i} - \ell_{4,i}^\intercal b(\tau) + \frac{1}{2}\upsilon_{6,i} b_i^2(\tau) + \upsilon_{8,i}\tilde{c}_i(\tau) + \ell_{5,i}^\intercal b(\tau)\tilde{c}_i(\tau) && \text{for } i = 1, \ldots, n_X \\
\partial_\tau \tilde{c}_i =& 2\upsilon_{2,i} - 2\upsilon_{9,i}\tilde{c}_i(\tau) + \upsilon_{5,i}\tilde{c}_i^2(\tau) && \text{for } i = 1, \ldots, n_X
\end{aligned}$$

with boundary conditions $a(0)$, $b_i(0)$ and $\tilde{c}_i(0)$ for $i = 1, \ldots, n_X$. Scalar parameter υ_0, vector parameters $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_6, w_8, w_9$ of size n_X, symmetric matrix parameters ℓ_5, ℓ_7 of size $n_X \times n_X$ and lower triangular matrix parameter ℓ_4 of size $n_X \times n_X$ correspond, in matrix notation, to the following decomposition

$$\begin{aligned}
R_t =&\ \upsilon_0 &+&\ \upsilon_1^\intercal X &+&\ \upsilon_2^\intercal X^2 \\
B_t =&\ \upsilon_3 &-&\ \ell_4^\intercal X \\
C_t =&\ \ell_5 &+&\ \operatorname{diag}(\upsilon_6 \odot X) \\
D_t =&\ \ell_7 &+&\ \operatorname{diag}(w_8 \odot X) &+&\ \operatorname{diag}\left(w_9 \odot X^2\right),
\end{aligned}$$

which is subject to the restrictions

$$\upsilon_6 \odot \tilde{\upsilon}_2 = 0 \tag{A.45}$$
$$\tilde{\upsilon}_2\tilde{\upsilon}_2^\intercal \odot \ell_5 = \operatorname{diag}(\tilde{\upsilon}_2 \odot \upsilon_5) \tag{A.46}$$
$$\tilde{\upsilon}_6 \mathbf{1}^\intercal \odot \ell_4 = \operatorname{diag}(\tilde{\upsilon}_6 \odot \upsilon_4) \tag{A.47}$$
$$\tilde{\upsilon}_2 \mathbf{1}^\intercal \odot \ell_4^\intercal = \operatorname{diag}(\tilde{\upsilon}_2 \odot \upsilon_4) \tag{A.48}$$

and, when ℓ_4 is a non-diagonal matrix, there is an additional restriction on the following expression to be a lower triangular matrix

$$\tilde{\upsilon}_2 \mathbf{1}^\intercal \odot \ell_5. \tag{A.49}$$

Here and in the following I use the shorthand notation

$$\begin{aligned}
\upsilon_4 \coloneqq& \operatorname{diag}(\ell_4) \\
\upsilon_5 \coloneqq& \operatorname{diag}(\ell_5) \\
\tilde{\upsilon}_6 \coloneqq& \mathbb{1}_{\upsilon_6 \neq 0} \\
\tilde{\upsilon}_2 \coloneqq& \mathbb{1}_{\upsilon_2 \neq 0 \,\vee\, \tilde{c}(0) \neq 0} \\
\upsilon_8 \coloneqq& \upsilon_3 + \frac{w_8}{2} \\
\upsilon_9 \coloneqq& \upsilon_4 - \frac{w_9}{2}.
\end{aligned}$$

The diagonalized ODEs above admit a more general decomposition of D, although it requires the use of tensor notation

$$\begin{aligned}
D^k{}_l =&\ (\ell_7)^k{}_l + {}_D\beta^k{}_l{}^i\, X_i + {}_D\omega^k{}_l{}^i{}_j\, X_i\, X^j \\
(w_8)_i \coloneqq&\ {}_D\beta_{iii} \\
(w_9)_i \coloneqq&\ {}_D\omega_{iiii}
\end{aligned} \tag{A.50}$$

subject to restrictions

$$(\tilde{v}_2)^k\,(\tilde{v}_2)^i\,{}_D\beta^{kki} = \delta^{ki}\,(\tilde{v}_2)^i\,(w_8)^i$$
$$(\tilde{v}_2)^k\,(\tilde{v}_2)^i\,(\tilde{v}_2)_j\,{}_D\omega^{kki}{}_j = \delta^{ki}\,\delta^i{}_j\,(\tilde{v}_2)^i\,(w_9)^i\,.$$

Section A.22 describes how to solve these diagonalized Riccati ODEs explicitly, which in turn solve the PDE as

$$g(t,X) = e^{a(T-t)+b(T-t)^\intercal X+\frac{1}{2}\tilde{c}(T-t)^\intercal X^2}.$$

The diagonalized Riccati ODEs were derived by imposing some restrictions to the main Riccati ODEs above. This diagonalization corresponds to the case in which $\eta = I$ is the identity matrix, ${}_B\omega^{ki}{}_j = 0$, ${}_C\omega^k{}_l{}^i{}_j = 0$, vectors $\upsilon_{6,i}, \upsilon_{2,i}$ satisfy the following diagonal constraints

$$\begin{aligned}{}_C\beta^k{}_l{}^i &= \delta^{ki}\,\delta_l{}^i\,(\upsilon_6)^i\\ {}_R\omega^i{}_j &= \delta^i{}_j\,(\upsilon_2)^i,\end{aligned}$$

and remaining terms correspond to

$$\begin{aligned}
&\upsilon_0 = {}_R\alpha && \upsilon_{1,i} = {}_R\beta^i && \upsilon_{3,i} = {}_B\alpha^i && (\ell_4)^k{}_p = -\,{}_B\hat{\beta}^k{}_p = -\,{}_B\beta_p{}^k\\
&\upsilon_{4,i} = \operatorname{diag}(\ell_4)^i && (\ell_5)^k{}_l = {}_C\alpha^k{}_l && \upsilon_{5,i} = \operatorname{diag}(\ell_5)^i && (\ell_7)^k{}_l = {}_D\alpha^k{}_l\\
&(w_8)_i = {}_D\beta_{iii} && (w_9)_i = {}_D\omega_{iiii} && \upsilon_{8,i} = \upsilon_{3,i} + \frac{w_{8,i}}{2} && \upsilon_{9,i} = \upsilon_{4,i} - \frac{w_{9,i}}{2}\\
& && && \tilde{\upsilon}_2 = \mathbb{1}_{\upsilon_2\neq 0\,\vee\,\tilde{c}(0)\neq 0} && \tilde{\upsilon}_6 = \mathbb{1}_{\upsilon_6\neq 0}.
\end{aligned}$$

The restriction below diagonalizes interactions of term ${}_C\alpha^k{}_l$ in (A.40) and prevents the introduction of reciprocal dependencies through ${}_C\alpha^k{}_l$ in (A.39)

$$(\tilde{v}_2)^i\,(\tilde{v}_2)_j\,(\ell_5)^i{}_j = \delta^i{}_j\,(\tilde{v}_2)^i\,(\upsilon_5)^i$$

giving (A.46). The following restrictions make the k, l traced diagonals of ${}_D\beta^k{}_l{}^i$ in (A.40) and of ${}_D\omega^k{}_l{}^i{}_j$ in (A.39) to become diagonal also with respect to i, j

$$(\tilde{v}_2)^k\,(\tilde{v}_2)^i\,{}_D\beta^{kki} = \delta^{ki}\,(\tilde{v}_2)^i\,(w_8)^i$$
$$(\tilde{v}_2)^k\,(\tilde{v}_2)^i\,(\tilde{v}_2)_j\,{}_D\omega^{kki}{}_j = \delta^{ki}\,\delta_j{}^i\,(\tilde{v}_2)^i\,(w_9)^i$$

giving the restrictions associated to decomposition (A.50). This other restriction ensures that constraint (A.43) is satisfied

$$(\upsilon_6)^i\,(\tilde{v}_2)^i = \mathbf{0}^i$$

giving rise to condition (A.45). The restriction

$$(\tilde{v}_6)^p\,(\ell_4)^p{}_k = \delta^p{}_k\,(\tilde{v}_6)^p\,(\upsilon_4)^p$$

makes sure that when $\tilde{v}_{6,i} \neq 0$, term ${}_{B,\tau}\beta^{ki}$ for $b_i(\tau)$ in (A.39) only have terms referencing component i giving rise to (A.47). The restriction

$$(\tilde{v}_2)^k\,(\ell_4)_p{}^k = \delta_p{}^k\,(\tilde{v}_2)^k\,(\upsilon_4)^k$$

helps to ensure a diagonal structure for $c_i(\tau)$ in (A.40) by imposing a partial diagonalization of ${}_{B,\tau}\hat{\beta}_j{}^\nu$ and gives rise to (A.48). To avoid reciprocal dependencies across ℓ_4, ℓ_5 terms in (A.39) when ℓ_4 is non-diagonal, I restrict the following expression to be a lower triangular matrix

$$(\tilde{v}_2)^k\,(\ell_5)^k{}_l$$

making non-diagonal matrix terms produce a lower triangular matrix expression

$$\left({}_{B,\tau}\beta_l{}^i + \tilde{c}(\tau)^i\,{}_{C,\tau}\alpha^i{}_l\right)\,b(\tau)^l$$

giving rise to (A.49).

A.22 Closed form solutions to diagonalized Riccati ODEs, Lemma 11

As shown in Section A.21, if the parameters of the semi-linear PDE from Definition 3 can be reformulated in terms of constant quadratic coefficients satisfying [4]

$$\begin{aligned}
R_t &= \upsilon_0 &+& \quad \upsilon_1^\intercal X &+& \quad \upsilon_2^\intercal X^2 \\
B_t &= \upsilon_3 &-& \quad \ell_4^\intercal X \\
C_t &= \ell_5 &+& \quad \operatorname{diag}(\upsilon_6 \odot X) \\
D_t &= \ell_7 &+& \quad \operatorname{diag}(w_8 \odot X) &+& \quad \operatorname{diag}\left(w_9 \odot X^2\right)
\end{aligned}$$

with scalar parameter υ_0, vector parameters $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_6, w_8, w_9$ of size n_X, symmetric matrix parameters ℓ_5, ℓ_7 of size $n_X \times n_X$ and lower triangular matrix parameter ℓ_4 of size $n_X \times n_X$ subject to restrictions

$$\upsilon_6 \odot \tilde{\upsilon}_2 = 0 \tag{A.51}$$

$$\tilde{\upsilon}_2 \tilde{\upsilon}_2^\intercal \odot \ell_5 = \operatorname{diag}(\tilde{\upsilon}_2 \odot \upsilon_5) \tag{A.52}$$

$$\tilde{\upsilon}_6 \mathbf{1}^\intercal \odot \ell_4 = \operatorname{diag}(\tilde{\upsilon}_6 \odot \upsilon_4) \tag{A.53}$$

$$\tilde{\upsilon}_2 \mathbf{1}^\intercal \odot \ell_4^\intercal = \operatorname{diag}(\tilde{\upsilon}_2 \odot \upsilon_4) \tag{A.54}$$

and subject to the boundary condition decomposition below with deterministic $a(0)$, $b(0)$, $\tilde{c}(0)$

$$g(T, X) = e^{a(0) + b(0)^\intercal X + \frac{1}{2}\tilde{c}(0)^\intercal X^2}$$

then the candidate solution corresponds to

$$g(t, X) = e^{a(T-t) + b(T-t)^\intercal X + \frac{1}{2}\tilde{c}(T-t)^\intercal X^2}$$

with $a(\tau), b(\tau), \tilde{c}(\tau)$ as solutions to the diagonalized Riccati ODEs below (A.56) (A.57) (A.58).

Here and in the following I use the shorthand notation

$$\begin{aligned}
\upsilon_4 &:= \operatorname{diag}(\ell_4) \\
\upsilon_5 &:= \operatorname{diag}(\ell_5) \\
\tilde{\upsilon}_6 &:= \mathbb{1}_{\upsilon_6 \neq 0} \\
\tilde{\upsilon}_2 &:= \mathbb{1}_{\upsilon_2 \neq 0 \,\vee\, \tilde{c}(0) \neq 0} \\
\upsilon_8 &:= \upsilon_3 + \frac{w_8}{2} \\
\upsilon_9 &:= \upsilon_4 - \frac{w_9}{2}.
\end{aligned}$$

When ℓ_4 is a non-diagonal matrix, there is an additional restriction on the following expression to be a lower triangular matrix

$$\tilde{\upsilon}_2 \mathbf{1}^\intercal \odot \ell_5. \tag{A.55}$$

Now consider the associated system of ODEs with constant parameters

$$\partial_\tau a = \upsilon_0 + \upsilon_3^\intercal b(\tau) + \frac{1}{2}\operatorname{tr}\left(\ell_5 b(\tau) b(\tau)^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\ell_7 \operatorname{diag}(\tilde{c}(\tau))\right) \tag{A.56}$$

$$\partial_\tau b_i = \upsilon_{1,i} - \ell_{4,i}^\intercal b(\tau) + \frac{1}{2}\upsilon_{6,i} b_i^2(\tau) + \upsilon_{8,i}\tilde{c}_i(\tau) + \ell_{5,i}^\intercal b(\tau)\tilde{c}_i(\tau) \qquad \text{for } i = 1, \ldots, n_X \tag{A.57}$$

$$\partial_\tau \tilde{c}_i = 2\upsilon_{2,i} - 2\upsilon_{9,i}\tilde{c}_i(\tau) + \upsilon_{5,i}\tilde{c}_i^2(\tau) \qquad \text{for } i = 1, \ldots, n_X \tag{A.58}$$

[4]Note that D also accepts a more general decomposition described in Section A.21 at the expense of more complex notation.

and constant boundaries $a(0)$, $b_i(0)$ and $\tilde{c}_i(0)$ for $i = 1, \ldots, n_X$.

The solution to $a(\tau)$ is

$$\begin{aligned} a(\tau) =& a(0) + \upsilon_0 \tau + \upsilon_3^\intercal \int_0^\tau b(s)\,\mathrm{d}s + \frac{1}{2}\operatorname{tr}\left(\ell_5 \int_0^\tau b(s)b(s)^\intercal\,\mathrm{d}s\right) \\ &+ \frac{1}{2}\operatorname{tr}\left(\ell_7 \operatorname{diag}\left(\int_0^\tau \tilde{c}(s)\,\mathrm{d}s\right)\right) \end{aligned} \tag{A.59}$$

The remaining parts of this section show how to explicitly solve the $b_i(\tau)$ and $\tilde{c}_i(\tau)$ for each i^{th} component under different restrictions. These solutions enter $a(\tau)$ again through the integrals in right hand side of (A.59). Closed form expressions for the integrals of $\tilde{c}_i(\tau)$, $b_i(\tau)$, $b_i^2(\tau)$ and $b_i(\tau)b_j(\tau)$ can be obtained in many cases, for instance through computer algebra systems like Mathematica, but they are not provided here. In my experience, off-diagonal elements $b_i(\tau)b_j(\tau)$ are the most troublesome although their integrals are not necessary when ℓ_5 is a diagonal matrix. When closed form integrals are not available, they can be computed numerically.

Solution when $\{\upsilon_{2,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{1,i} = 0$ and $b_i(0) = 0$ and $\ell_{4,i,-i} = 0\}$ In this case we have that $b_i(\tau) = 0$ and $\tilde{c}_i(\tau) = 0$ thus these components disappear and their associated integrals disappear from (A.59).

Solution when $\{\upsilon_{2,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{1,i} \neq 0$ or $b_i(0) \neq 0$ or $\ell_{4,i,-i} \neq 0\}$ and $\upsilon_{6,i} = 0$ In this case we have that $\tilde{c}_i(\tau) = 0$ and (A.57) transforms into a simple linear ODE for $b_i(\tau)$ with solution

$$b_i(\tau) = \begin{cases} b_i(0) + \upsilon_{1,i}\tau - \ell_{4,i,-i}^\intercal \int_0^\tau b_{-i}(s)\,\mathrm{d}s & \text{if } \upsilon_{4,i} = 0 \\ b_i(0)e^{-\upsilon_{4,i}\tau} + \upsilon_{1,i}\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} - \ell_{4,i,-i}^\intercal \int_0^\tau b_{-i}(s)e^{-\upsilon_{4,i}(\tau-s)}\,\mathrm{d}s & \text{otherwise.} \end{cases}$$

Negative indexes like $b_{-i}(s)$ select all the positions along the axis except for the specified one i. Restrictions (A.52) (A.53) (A.55) prevent simultaneous dependencies among $b_{-i}(\tau)$ components. When dealing only with this kind of components, the restriction of ℓ_4 to be a lower triangular matrix makes it possible to sequentially solve each $b_i(\tau)$ component in order $i = 1, \ldots, n_X$.

Solution when $\{\upsilon_{2,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{1,i} \neq 0$ or $b_i(0) \neq 0\}$ and $\upsilon_{6,i} \neq 0$ In this instance, we have that $\tilde{c}_i(\tau) = 0$ and we focus on b_i. Solutions for $b_i(\tau)$ fall into different cases depending on $\varkappa_i$

$$\varkappa_i = \upsilon_{4,i}^2 - 2\upsilon_{1,i}\upsilon_{6,i}.$$

Rearrange (A.57) to integrate over time horizon

$$\int_0^\tau 1\,\mathrm{d}s = \int_0^\tau \frac{\partial_s b_i(s)}{\upsilon_{1,i} - \upsilon_{4,i}b_i(s) + \frac{1}{2}\upsilon_{6,i}b_i^2(s)}\,\mathrm{d}s$$

The solution can be found using integration tables

$$
b_i(\tau) = \begin{cases}
\dfrac{\left(\upsilon_{1,i}+\frac{\left(-\upsilon_{4,i}-\sqrt{\varkappa_i}\right)}{2}b_i(0)\right)\left(1-e^{-\sqrt{\varkappa_i}\tau}\right)+\sqrt{\varkappa_i}b_i(0)}{\sqrt{\varkappa_i}-\frac{\left(\upsilon_{6,i}b_i(0)-\upsilon_{4,i}+\sqrt{\varkappa_i}\right)}{2}\left(1-e^{-\sqrt{\varkappa_i}\tau}\right)} & \text{if } \varkappa_i > 0 \text{ and } \left\{\begin{array}{c} \sqrt{\varkappa_i} > \upsilon_{6,i}b_i(0) - \upsilon_{4,i} \\ \text{or else} \\ \tau < -\dfrac{\log\left(\frac{\upsilon_{6,i}b_i(0)-\upsilon_{4,i}-\sqrt{\varkappa_i}}{\upsilon_{6,i}b_i(0)-\upsilon_{4,i}+\sqrt{\varkappa_i}}\right)}{\sqrt{\varkappa_i}} \end{array}\right\} \\
\dfrac{\upsilon_{4,i}-\frac{\upsilon_{4,i}-\upsilon_{6,i}b_i(0)}{1+\frac{\upsilon_{4,i}-\upsilon_{6,i}b_i(0)}{2}\tau}}{\upsilon_{6,i}} & \text{if } \varkappa_i = 0 \text{ and } \left\{\begin{array}{l} 0 \le \upsilon_{4,i} - \upsilon_{6,i}b_i(0) \\ \text{or else} \\ \tau < -\dfrac{2}{\upsilon_{4,i} - \upsilon_{6,i}b_i(0)} \end{array}\right\} \\
\dfrac{\upsilon_{4,i}+\sqrt{-\varkappa_i}\tan\left(\arctan\left(\frac{\upsilon_{6,i}b_i(0)-\upsilon_{4,i}}{\sqrt{-\varkappa_i}}\right)+\frac{\sqrt{-\varkappa_i}}{2}\tau\right)}{\upsilon_{6,i}} & \text{if } \left\{\begin{array}{l} \varkappa_i < 0 \\ \text{and } \tau < \dfrac{\pi - 2\arctan\left(\frac{\upsilon_{6,i}b_i(0)-\upsilon_{4,i}}{\sqrt{-\varkappa_i}}\right)}{\sqrt{-\varkappa_i}} \end{array}\right\}
\end{cases} . \tag{A.60}
$$

Note that π refers to the trigonometric constant in this context. Dependencies on other $b(\tau)$ components are ruled out by restrictions (A.51) (A.53).

Solution when $\{\upsilon_{2,i} \neq 0$ or $\tilde{c}_i(0) \neq 0\}$ and $\upsilon_{6,i} = 0$ In this instance, solutions for $\tilde{c}_i(\tau)$ fall into different cases depending on $\hbar_i$ as detailed by Kim and Omberg (1996)

$$
\hbar_i = 4\upsilon_{9,i}^2 - 8\upsilon_{5,i}\upsilon_{2,i}.
$$

First we rearrange (A.58) and integrate over time horizon

$$
\int_0^\tau 1 \, \mathrm{d}s = \int_0^\tau \frac{\partial_s \tilde{c}_i(s)}{2\upsilon_{2,i} - 2\upsilon_{9,i}\tilde{c}_i(s) + \upsilon_{5,i}\tilde{c}_i^2(s)} \, \mathrm{d}s
$$

then we can find the solution using integration tables

$$
\tilde{c}_i(\tau) = \begin{cases}
2\upsilon_{2,i}\tau + \tilde{c}_i(0) & \text{if } \upsilon_{5,i} = 0 \text{ and } \upsilon_{9,i} = 0 \\
\upsilon_{2,i}\frac{1-e^{-2\upsilon_{9,i}\tau}}{\upsilon_{9,i}} + \tilde{c}_i(0)e^{-2\upsilon_{9,i}\tau} & \text{if } \upsilon_{5,i} = 0 \text{ and } \upsilon_{9,i} \neq 0 \\
\dfrac{\left(2\upsilon_{2,i}+\left(-\upsilon_{9,i}-\frac{\sqrt{\hbar_i}}{2}\right)\tilde{c}_i(0)\right)\left(1-e^{-\sqrt{\hbar_i}\tau}\right)+\sqrt{\hbar_i}\tilde{c}_i(0)}{\sqrt{\hbar_i}-\left(\upsilon_{5,i}\tilde{c}_i(0)-\upsilon_{9,i}+\frac{\sqrt{\hbar_i}}{2}\right)\left(1-e^{-\sqrt{\hbar_i}\tau}\right)} & \text{if } \left\{\begin{array}{l} \upsilon_{5,i} \neq 0 \text{ and } \hbar_i > 0 \\ \text{and } \left\{\begin{array}{c} \upsilon_{9,i} > \upsilon_{5,i}\tilde{c}_i(0) - \dfrac{1}{2}\sqrt{\hbar_i} \\ \text{or else} \\ \tau < -\dfrac{\log\left(\frac{\upsilon_{5,i}\tilde{c}_i(0)-\upsilon_{9,i}-\frac{\sqrt{\hbar_i}}{2}}{\upsilon_{5,i}\tilde{c}_i(0)-\upsilon_{9,i}+\frac{\sqrt{\hbar_i}}{2}}\right)}{\sqrt{\hbar_i}} \end{array}\right\} \end{array}\right\} \\
\dfrac{\upsilon_{9,i}-\frac{\upsilon_{9,i}-\upsilon_{5,i}c_i(0)}{1+\left(\upsilon_{9,i}-\upsilon_{5,i}c_i(0)\right)\tau}}{\upsilon_{5,i}} & \text{if } \left\{\begin{array}{l} \upsilon_{5,i} \neq 0 \text{ and } \hbar_i = 0 \\ \text{and } \left\{\begin{array}{l} 0 \le \upsilon_{9,i} - \upsilon_{5,i}c_i(0) \\ \text{or else} \\ \tau < -\dfrac{1}{\upsilon_{9,i} - \upsilon_{5,i}c_i(0)} \end{array}\right\} \end{array}\right\} \\
\dfrac{\upsilon_{9,i}+\frac{\sqrt{-\hbar_i}}{2}\tan\left(\arctan\left(\frac{2\upsilon_{5,i}\tilde{c}_i(0)-2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right)+\frac{\sqrt{-\hbar_i}}{2}\tau\right)}{\upsilon_{5,i}} & \text{if } \left\{\begin{array}{l} \upsilon_{5,i} \neq 0 \text{ and } \hbar_i < 0 \\ \text{and } \tau < \dfrac{\pi - 2\arctan\left(\frac{2\upsilon_{5,i}\tilde{c}_i(0)-2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right)}{\sqrt{-\hbar_i}} \end{array}\right\}
\end{cases} . \tag{A.61}
$$

Once $\tilde{c}_i(\tau)$ is solved, the expression can be plugged into the ODE for $b_i(\tau)$ (A.57). Solving this inhomogenous linear ODE is straightforward as long as it remains finite

$$b_i(\tau) = \frac{\int_0^\tau e^{\int_0^s (\upsilon_{4,i}-\upsilon_{5,i}\tilde{c}_i(u))\mathrm{d}u} \left(\upsilon_{1,i} + \upsilon_{8,i}\tilde{c}_i(s) + \tilde{c}_i(s)\ell_{5,i,-i}^\intercal b_{-i}(s) - \ell_{4,i,-i}^\intercal b_{-i}(s)\right) \mathrm{d}s + b_i(0)}{e^{\int_0^\tau (\upsilon_{4,i}-\upsilon_{5,i}\tilde{c}_i(s))\mathrm{d}s}}.$$

Negative indexes like $b_{-i}(s)$ select all the positions along the axis except for the specified one i. Restrictions (A.52) (A.53) (A.55) prevent simultaneous dependencies among $b_{-i}(\tau)$ components. The restriction of ℓ_4 to be a lower triangular matrix together with restriction (A.55) makes it possible to sequentially solve each $b_i(\tau)$ component in order $i = 1, \ldots, n_X$. When ℓ_4 is a diagonal matrix, restriction (A.55) is not necessary. In this case one can solve first for each j component of the $b_{-i}(\tau)$ associated with non-zero $\ell_{5,i,-i}$ coefficients because their $\tilde{c}_j(s) = 0$ and therefore they cannot depend on other $b_{-j}(\tau)$ through $\ell_{5,i,-i}$.

Known closed form solutions to $b_i(\tau)$, assuming that $\ell_{5,i,-i} = 0$ and $\ell_{4,i,-i} = 0$, are displayed separately for the main groups of cases depending on the values of $\upsilon_{5,i}$ and $\hbar_i$.

If $\upsilon_{5,i} = 0$

$$b_i(\tau) = \begin{cases} b_i(0) + (\upsilon_{1,i} + \upsilon_{8,i}\tilde{c}_i(0))\,\tau + \upsilon_{8,i}\upsilon_{2,i}\tau^2 & \text{if } \upsilon_{9,i} = 0 \text{ and } \upsilon_{4,i} = 0 \\ b_i(0)e^{-\upsilon_{4,i}\tau} + 2\frac{\upsilon_{8,i}\upsilon_{2,i}}{\upsilon_{4,i}}\tau + \left(\upsilon_{1,i} + \upsilon_{8,i}\left(\tilde{c}_i(0) - 2\frac{\upsilon_{2,i}}{\upsilon_{4,i}}\right)\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} & \text{if } \upsilon_{9,i} = 0 \text{ and } \upsilon_{4,i} \neq 0 \\ b_i(0) + \left(\upsilon_{1,i} + \frac{\upsilon_{8,i}\upsilon_{2,i}}{\upsilon_{9,i}}\right)\tau + \upsilon_{8,i}\left(\tilde{c}_i(0) - \frac{\upsilon_{2,i}}{\upsilon_{9,i}}\right)\frac{1-e^{-2\upsilon_{9,i}\tau}}{2\upsilon_{9,i}} & \text{if } \upsilon_{9,i} \neq 0 \text{ and } \upsilon_{4,i} = 0 \\ \left(b_i(0)e^{-\upsilon_{4,i}\tau} + \left(\upsilon_{1,i} + 2\frac{\upsilon_{8,i}\upsilon_{2,i}}{\upsilon_{4,i}}\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} \right. \\ \qquad \left. + \upsilon_{8,i}\left(\tilde{c}_i(0) - \frac{2\upsilon_{2,i}}{\upsilon_{4,i}}\right)\tau e^{-\upsilon_{4,i}\tau}\right) & \text{if } 2\upsilon_{9,i} = \upsilon_{4,i} \neq 0 \\ \left(b_i(0)e^{-\upsilon_{4,i}\tau} + \left(\upsilon_{1,i} + \frac{\upsilon_{8,i}\upsilon_{2,i}}{\upsilon_{9,i}}\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} \right. \\ \qquad \left. + \upsilon_{8,i}\left(\frac{\upsilon_{2,i}}{\upsilon_{9,i}} - \tilde{c}_i(0)\right)\frac{e^{-2\upsilon_{9,i}\tau} - e^{-\upsilon_{4,i}\tau}}{2\upsilon_{9,i} - \upsilon_{4,i}}\right) & \text{if } \left\{\begin{matrix} \upsilon_{9,i} \neq 0 \text{ and } \upsilon_{4,i} \neq 0 \\ \text{and } 2\upsilon_{9,i} \neq \upsilon_{4,i} \end{matrix}\right\} \end{cases}$$

If $\upsilon_{5,i} \neq 0$ and $\hbar_i > 0$ and $\tilde{c}_i(0) = 0$

$$b_i(\tau) = \begin{cases} 2\upsilon_{2,i}\frac{\left(\frac{\sqrt{\hbar_i}}{2\upsilon_{2,i}} b_i(0) + \left(\upsilon_{8,i} + \left(\upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{2\upsilon_{2,i}}\right)\tau - \left(\upsilon_{8,i} + \left(\upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{2\upsilon_{2,i}}\right)\frac{1-e^{-\sqrt{\hbar_i}\tau}}{\sqrt{\hbar_i}}\right)}{\sqrt{\hbar_i} - \left(-\upsilon_{9,i} + \frac{1}{2}\sqrt{\hbar_i}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } \upsilon_{9,i} - \upsilon_{4,i} = \frac{\sqrt{\hbar_i}}{2} \\ 2\upsilon_{2,i}\frac{\left(\frac{\sqrt{\hbar_i}}{2\upsilon_{2,i}} b_i(0)e^{-\sqrt{\hbar_i}\tau} + \left(\upsilon_{8,i} + \left(\upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{2\upsilon_{2,i}}\right)\frac{1-e^{-\sqrt{\hbar_i}\tau}}{\sqrt{\hbar_i}} - \left(\upsilon_{8,i} + \left(\upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{2\upsilon_{2,i}}\right)\tau e^{-\sqrt{\hbar_i}\tau}\right)}{\sqrt{\hbar_i} - \left(-\upsilon_{9,i} + \frac{1}{2}\sqrt{\hbar_i}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } \upsilon_{9,i} - \upsilon_{4,i} = -\frac{\sqrt{\hbar_i}}{2} \\ 2\upsilon_{2,i}\frac{\left(K_{1,i} + K_{2,i}e^{-\sqrt{\hbar_i}\tau} + \left(b_i(0)\frac{\sqrt{\hbar_i}}{2\upsilon_{2,i}} - K_{1,i} - K_{2,i}\right)e^{\left(\upsilon_{9,i} - \upsilon_{4,i} - \frac{\sqrt{\hbar_i}}{2}\right)\tau}\right)}{\sqrt{\hbar_i} - \left(-\upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } |\upsilon_{9,i} - \upsilon_{4,i}| \neq \frac{\sqrt{\hbar_i}}{2} \end{cases}$$

under the restrictions

$$\upsilon_{9,i} > -\frac{\sqrt{\hbar_i}}{2} \qquad \text{or else} \qquad \tau < -\frac{\log\left(\frac{-2\upsilon_{9,i} - \sqrt{\hbar_i}}{-2\upsilon_{9,i} + \sqrt{\hbar_i}}\right)}{\sqrt{\hbar_i}}$$

where

$$K_{1,i} = \frac{2\upsilon_{8,i} + \left(\upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{\upsilon_{2,i}}}{\sqrt{\hbar_i} - 2\left(\upsilon_{9,i} - \upsilon_{4,i}\right)}$$
$$K_{2,i} = \frac{2\upsilon_{8,i} + \left(\upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{1,i}}{\upsilon_{2,i}}}{\sqrt{\hbar_i} + 2\left(\upsilon_{9,i} - \upsilon_{4,i}\right)}.$$

If $\upsilon_{5,i} \neq 0$ and $\hbar_i = 0$

$$
b_i(\tau) = \begin{cases}
\frac{b_i(0) + \left(\upsilon_{1,i} + \upsilon_{8,i}\frac{\upsilon_{4,i}}{\upsilon_{5,i}}\right)\left(\tau + \frac{\upsilon_{4,i}}{2}\tau^2\right)}{1+\upsilon_{4,i}\tau} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} = \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} = \upsilon_{5,i}c_i(0) \end{matrix}\right\} \\
\frac{\left(b_i(0) + \left(\upsilon_{1,i} + \upsilon_{8,i}\frac{\upsilon_{4,i}}{\upsilon_{5,i}}\right)\left(\tau + \frac{\upsilon_{4,i}}{2}\tau^2\right) - \frac{\upsilon_{8,i}}{\upsilon_{5,i}}\left(\upsilon_{4,i} - \upsilon_{5,i}c_i(0)\right)\tau - c_i(0)\upsilon_{8,i}\left(\tau - \frac{\log(1+(\upsilon_{4,i}-\upsilon_{5,i}c_i(0))\tau)}{\upsilon_{4,i}-\upsilon_{5,i}c_i(0)}\right)\right)}{1+\upsilon_{4,i}\tau} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} = \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} \neq \upsilon_{5,i}c_i(0) \end{matrix}\right\} \\
\frac{\left(b_i(0) + \left(\upsilon_{1,i} + \frac{\upsilon_{8,i}}{\upsilon_{5,i}}\upsilon_{9,i}\right)\left(\frac{1-e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}}{\upsilon_{9,i}-\upsilon_{4,i}} - \upsilon_{9,i}\frac{\tau e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau} - \frac{1-e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}}{\upsilon_{9,i}-\upsilon_{4,i}}}{\upsilon_{9,i}-\upsilon_{4,i}}\right)\right)}{(1+\upsilon_{9,i}\tau)e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} \neq \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} = \upsilon_{5,i}c_i(0) \end{matrix}\right\} \\
\frac{\left(b_i(0) + \upsilon_{1,i}\frac{1-e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}}{\upsilon_{9,i}-\upsilon_{4,i}} - \frac{\left(\upsilon_{1,i}+\upsilon_{8,i}\frac{\upsilon_{9,i}}{\upsilon_{5,i}}\right)\upsilon_{9,i}}{\upsilon_{9,i}-\upsilon_{4,i}}\left(\tau e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau} - \frac{1-e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}}{\upsilon_{9,i}-\upsilon_{4,i}}\right) + c_i(0)\upsilon_{8,i}e^{\frac{\upsilon_{9,i}-\upsilon_{4,i}}{\upsilon_{9,i}-\upsilon_{5,i}c_i(0)}} \cdot \frac{\left(\mathrm{Ei}\left(\frac{-(\upsilon_{9,i}-\upsilon_{4,i})}{\upsilon_{9,i}-\upsilon_{5,i}c_i(0)} - (\upsilon_{9,i}-\upsilon_{4,i})\tau\right) - \mathrm{Ei}\left(\frac{-(\upsilon_{9,i}-\upsilon_{4,i})}{\upsilon_{9,i}-\upsilon_{5,i}c_i(0)}\right)\right)}{\upsilon_{9,i}-\upsilon_{5,i}c_i(0)}\right)}{(1+\upsilon_{9,i}\tau)e^{-(\upsilon_{9,i}-\upsilon_{4,i})\tau}} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} \neq \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} \neq \upsilon_{5,i}c_i(0) \end{matrix}\right\}
\end{cases}
$$

under the restrictions

$$\upsilon_{9,i} \geq 0 \qquad \text{or else} \qquad \tau < -\frac{1}{\upsilon_{9,i}}.$$

where Ei is the exponential integral $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,\mathrm{d}t$.

If $\upsilon_{5,i} \neq 0$ and $\hbar_i < 0$

$$
\begin{aligned}
b_i(\tau) =& b_i(0)e^{-(\upsilon_{4,i}-\upsilon_{9,i})\tau}\sec\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right)\cos\left(K_{3,i}\right) \\
&+ \frac{\left(K_{4,i} + K_{5,i}\tan\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right) - e^{-(\upsilon_{4,i}-\upsilon_{9,i})\tau}\sec\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right)\left(\begin{matrix} K_{4,i}\cos\left(K_{3,i}\right) \\ + K_{5,i}\sin\left(K_{3,i}\right) \end{matrix}\right)\right)}{\left(\upsilon_{4,i}-\upsilon_{9,i}\right)^2 - \frac{\hbar_i}{4}}
\end{aligned}
$$

under the restriction

$$\tau < \frac{\pi - 2\arctan\left(\frac{2\upsilon_{5,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right)}{\sqrt{-\hbar_i}}$$

where

$$\begin{aligned}
K_{3,i} =& \arctan\left(\frac{2\upsilon_{5,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right) \\
K_{4,i} =& \upsilon_{1,i}\left(\upsilon_{4,i} - \upsilon_{9,i}\right) + \upsilon_{8,i}\left(\frac{\upsilon_{9,i}\upsilon_{4,i}}{\upsilon_{5,i}} - 2\upsilon_{2,i}\right) \\
K_{5,i} =& \left(\upsilon_{1,i} + \frac{\upsilon_{8,i}}{\upsilon_{5,i}}\upsilon_{4,i}\right)\frac{\sqrt{-\hbar_i}}{2}.
\end{aligned}$$

Closed form expressions for integrals of $\tilde{c}_i$, b_i and b_i^2 can be obtained for many cases. Imposing restrictions like $\upsilon_{9,i} = \upsilon_{4,i}$ and $\tilde{c}_i(0) = b_i(0) = 0$ helps to find closed form expressions for tough cases.

Solution when $\{\upsilon_{2,i} \neq 0$ **or** $\tilde{c}_i(0) \neq 0\}$ **and** $\{\upsilon_{1,i} = 0$ **and** $b_i(0) = 0\}$ **and** $\upsilon_{8,i} = 0$ **and** $\ell_{5,i,-i} = 0$ **and** $\ell_{4,i,-i} = 0$ Then $b_i(\tau) = 0$ and $\tilde{c}_i(\tau)$ coincides with (A.61).

Solution when $\{\upsilon_{2,i} \neq 0$ **or** $\tilde{c}_i(0) \neq 0\}$ **and** $\upsilon_{6,i} \neq 0$ **and** $\ell_{5,i} = 0$ **and** $\upsilon_{8,i} = 0$ Then $b_i(\tau)$ and $\tilde{c}_i(\tau)$ coincide with restricted instances of (A.60) and (A.61). Although they technically solve (A.57) (A.58), these constraints do not correspond to quadratic cases from Section A.21.

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